Math 19: Calculus **Summer 2010**

Practice Problems on Limits and Continuity

1 A tank contains 10 liters of pure water. Salt water containing 20 grams of salt per liter is pumped into the tank at 2 liters per minute.

1. Express the salt concentration $C(t)$ after t minutes (in g/L).
2. What is the long-term concentration of salt, i.e., $\lim_{t\to\infty} C(t)$?

Solution:

1. The concentration is, in units of g/L,

$$C(t) = \frac{\text{total salt}}{\text{total volume}} = \frac{20 \cdot 2 \cdot t}{10 + 2 \cdot t} = \boxed{\frac{20t}{5+t}}$$

2. The long-term concentration is, in units of g/L,

$$\lim_{t\to+\infty} \frac{20t}{5+t} = \lim_{t\to+\infty} \frac{20t}{5+t} \cdot \frac{1/t}{1/t} = \lim_{t\to+\infty} \frac{20}{5/t+1} = \boxed{20}$$

2 Find the values of a and b that make $f(x)$ continuous for all real x.

$$f(x) = \begin{cases} be^x + a + 1, & x \leq 0 \\ ax^2 + b(x+3), & 0 < x \leq 1 \\ a\cos(\pi x) + 7bx, & x > 1 \end{cases}$$

Solution: We note that the functions are continuous on their domains, so we check that the left- and right-hand limits agree at the boundary x-values. At $x = 0$,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} be^x + a + 1 = b + a + 1,$$
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} ax^2 + b(x+3) = 3b,$$

so $b + a + 1 = 3b$, and $a = 2b - 1$. Next, at $x = 1$,

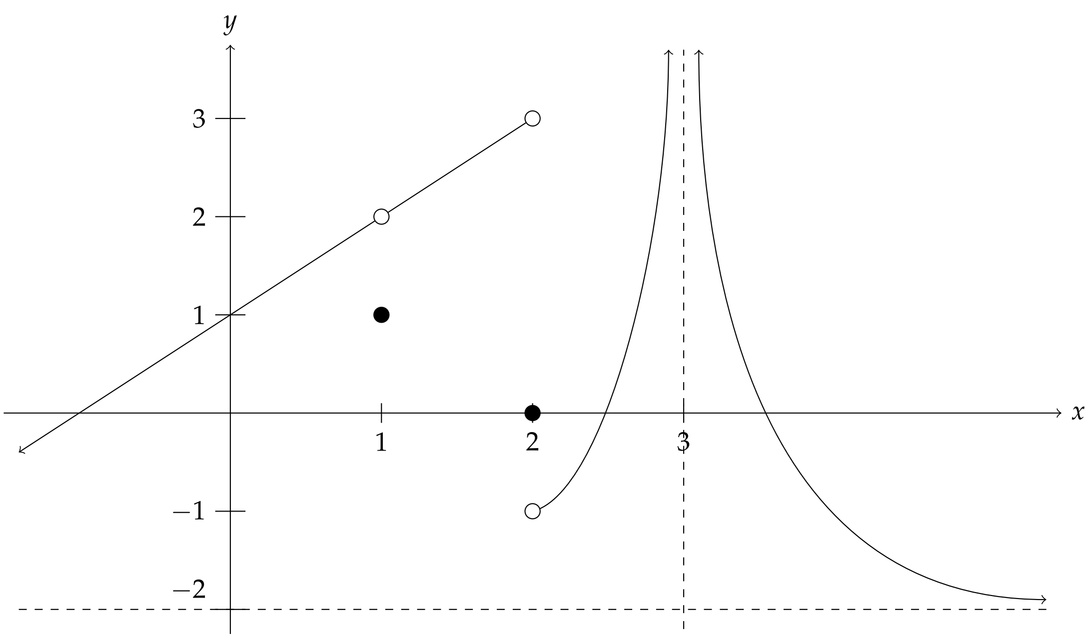
$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} ax^2 + b(x+3) = a + 4b,$$
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} a\cos(\pi x) + 7bx = -a + 7b,$$

so $a + 4b = -a + 7b$, and $2a = 3b$. Solving this linear system in a and b yields $\boxed{b = 2 \text{ and } a = 3}$ as the only solution.

3 Sketch the graph of a function f with the following properties:

- $\lim_{x\to 1} f(x) = 2$, but $f(1) = 1$
- $\lim_{x\to 3} f(x) = +\infty$
- $\lim_{x\to 2^+} f(x) = -1$, $\lim_{x\to 2^-} f(x) = 3$
- $\lim_{x\to +\infty} f(x) = -2$
- $\lim_{x\to -\infty} f(x) = -\infty$

Solution: Answers may vary, but here is a representative solution:



4 Show that the equation $\sqrt{x-5} = \dfrac{1}{x+3}$ has at least one real solution.

Solution: Let $f(x) = \sqrt{x-5} - \dfrac{1}{x+3}$, so that $f(x) = 0$ if and only if x is a solution to the equation. Then f is defined and continuous for all $x \geq 5$. Evaluating f at 5 and at 6, we see that

$$f(5) = \sqrt{5-5} - \frac{1}{5+3} = -\frac{1}{8} < 0 \qquad \text{and} \qquad f(6) = \sqrt{6-5} - \frac{1}{6+3} = \frac{8}{9} > 0.$$

By the Intermediate Value Theorem, there is some c in the interval $(5, 6)$ so that $f(c) = 0$, so f has at least one root.

(In fact, it is possible to reduce this equation to the cubic polynomial equation $(x-5)(x+3)^2 - 1 = 0$, and it is unpleasant but not impossible to find its roots exactly; the only valid root of the original equation is

$$c = -\frac{1}{3} + \frac{1}{3}\sqrt[3]{\frac{1051 - 15\sqrt{249}}{2}} + \frac{1}{3}\sqrt[3]{\frac{1051 + 15\sqrt{249}}{2}} \approx 5.01556\ldots.)$$

5 Consider the rational function

$$f(x) = \frac{x^5 - x^4 - 2x^3}{x^4 - 3x^3 - x^2 + 3x}$$

- For what values of a does f have a removable discontinuity at a? What is $\lim_{x\to a} f(x)$ at those a?
- For what values of a does f have an infinite discontinuity at a?
- What is $\lim_{x\to+\infty} f(x)$?

(Hint: Factor the numerator and the denominator.)

Solution: We factor the numerator and denominator of f to obtain

$$f(x) = \frac{x^3(x+1)(x-2)}{x(x+1)(x-1)(x-3)}.$$

Away from $x = 0$ and $x = -1$, $f(x)$ simplifies to

$$\frac{x^2(x-2)}{(x-1)(x-3)},$$

which is defined and continuous at these x-values. Thus, f has removable discontinuities at $x = 0$ and at $x = -1$. From this form of f, we also compute that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x^2(x-2)}{(x-1)(x-3)} = \frac{0^2(-2)}{(-1)(-3)} = 0 \quad \text{and}$$
$$\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{x^2(x-2)}{(x-1)(x-3)} = \frac{(-1)^2(-3)}{(-2)(-4)} = -\frac{3}{8}.$$

At $x = 1$ and at $x = 3$, however, the numerator of the original function is not 0, so f has infinite singularities (and vertical asymptotes) at these x-values.

Finally, we observe that

$$\lim_{x\to+\infty} \frac{x^5 - x^4 - 2x^3}{x^4 - 3x^3 - x^2 + 3x} = \lim_{x\to+\infty} \frac{x - 1 - \frac{2}{x}}{1 - \frac{3}{x} - \frac{1}{x^2} + \frac{3}{x^3}} = +\infty.$$

6 Find the value of a such that

$$\lim_{x \to -1} \frac{2x^2 - ax - 14}{x^2 - 2x - 3}$$

exists. What is the value of the limit?

Solution: We observe that $\lim_{x \to -1} x^2 - 2x - 3 = 1 + 2 - 3 = 0$, so in order for this limit to exist, we need the limit of the numerator as $x \to -1$ to be 0 as well. Since

$$\lim_{x \to -1} 2x^2 - ax - 14 = 2 + a - 14 = a - 12,$$

$a - 12 = 0$, and $\boxed{a = 12.}$ Then, away from $x = -1$,

$$\frac{2x^2 - 12x - 14}{x^2 - 2x - 3} = \frac{2(x-7)(x+1)}{(x-3)(x+1)} = \frac{2(x-7)}{(x-3)}.$$

As $x \to 1$, we see by evaluation that the limit is $\frac{2(-8)}{(-4)} = \boxed{4.}$