Math 19: Calculus **Summer 2010**

Background Material

This handout summarizes all the background material you are expected to know when you start Math 19. You will be quizzed on this material on Wednesday, June 30. Many of the formulas in this handout are also available on the reference page at the very beginning of the textbook.

1 Solving Equations and Inequalities

You should know how to solve equations and inequalities. This includes:

1. Avoiding division by zero. For example, to solve
$$x^3 - x = x,$$
you cannot divide everything by x and say that
$$x^2 - 1 = 1,$$
and so $x = \pm\sqrt{2}$. The reason you cannot do this is that x might equal 0, and in fact, $x = 0$ is a solution to the original equation. So instead of dividing by x, you collect everything to one side of the equation and then factor out an x:
$$\begin{aligned} x^3 - x &= x \\ x^3 - 2x &= 0 \\ x(x^2 - 2) &= 0 \\ x(x+\sqrt{2})(x-\sqrt{2}) &= 0 \end{aligned}$$
So we see $x = 0$ or $x = \pm\sqrt{2}$.

2. Solving equalities and inequalities with absolute values. For example:
$$\begin{aligned} &|x-3| = 4 \\ &x - 3 = 4 \quad \text{or} \quad x - 3 = -4 \\ &x = 7 \quad \text{or} \quad x = -1 \end{aligned}$$
$$\begin{aligned} &|x-3| \le 4 \\ &-4 \le x - 3 \le 4 \\ &-1 \le x \le 7 \end{aligned}$$
$$\begin{aligned} &|x-3| \ge 4 \\ &x - 3 \le -4 \quad \text{or} \quad x - 3 \ge 4 \\ &x \le -1 \quad \text{or} \quad x \ge 7 \end{aligned}$$

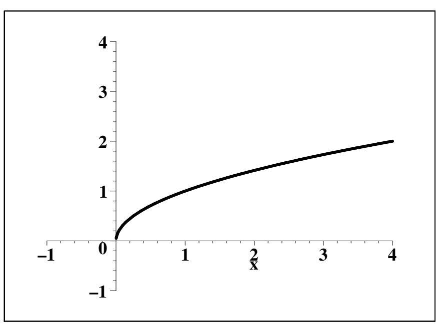
3. When solving equalities and inequalities, you have to be careful with things like square roots. For example:
   - $x^2 = 4$ has two solutions: $x = 2$, $x = -2$.
   - $x^2 \le 4$ has the solution $-2 \le x \le 2$; the answer $x \le 2$ is *incorrect*.
   - $x^2 \ge 4$ has the solution $x \le -2$ or $x \ge 2$; the answer $x \ge 2$ is *incorrect*.
4. Remember that when solving inequalities, if you divide by a negative number, you need to flip the inequality sign around:

$$\begin{aligned} -2x &\le 4 \\ x &\ge -2 \end{aligned}$$

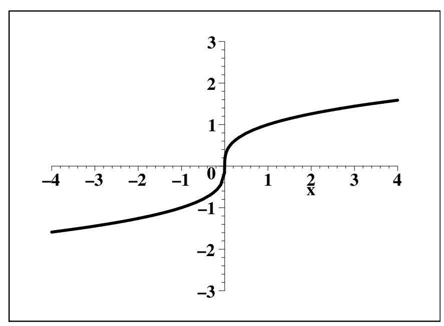
2 Simplifying Algebraic Expressions

1. You should know how to handle fractions (adding, multiplying, dividing, using common denominators, cancelling).
2. Here are the graphs of two root functions:

$f(x) = \sqrt{x}$



$f(x) = \sqrt[3]{x}$



You should know how to factor a perfect square out of a square root. For example:

$$\begin{aligned} \sqrt{18} &= 3\sqrt{2} \\ \sqrt{x^3 - x^2} &= x\sqrt{1 - x} \end{aligned}$$

3 Polynomials

1. The equation of a straight line with slope a and y-intercept b is $y = ax + b$.

   The equation of a straight line with slope a that goes through the point (x_0, y_0) is $y - y_0 = a(x - x_0)$.

   The equation of a straight line that goes through the points (x_0, y_0) and (x_1, y_1) is $y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$.

2. The y**-intercepts** of a curve are the y-values at which the curve intersects the y-axis. They can be found by letting $x = 0$ and solving for y.

   The x**-intercepts** of a curve are the x-values at which the curve intersects the x-axis. They can be found by letting $y = 0$ and solving for x.

3. The **zeroes** or **roots** of a function $y = f(x)$ are the values of x for which $y = f(x) = 0$; that is, they are the values of x where $y = f(x)$ crosses the x-axis, and so are the x-intercepts.

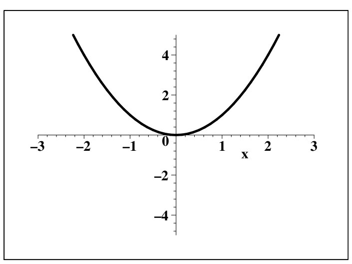
4. The **Quadratic Formula** says that the quadratic polynomial $ax^2 + bx + c$ has roots

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

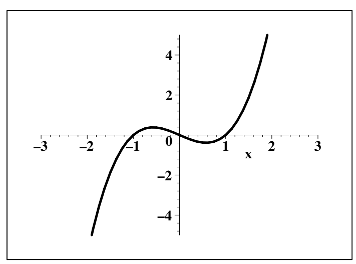
5. A **linear polynomial** has the form $y = ax + b$; it is the equation of a straight line. A **quadratic polynomial** has the form $y = ax^2 + bx + c$; it is the equation of a parabola. A **cubic polynomial** has the form $y = ax^3 + bx^2 + cx + d$; it is the equation of a cubic curve.

6. The graphs of typical quadratic, cubic, and quartic polynomials look like

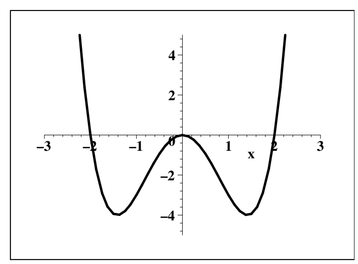
$f(x) = x^2$



$f(x) = x^3 - x$



$f(x) = x^4 - 4x^3 + 4x^2$

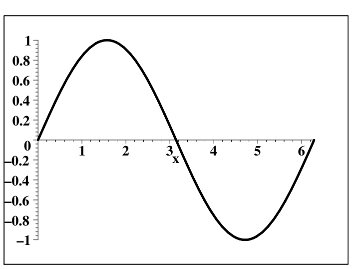


4 Trigonometric Functions

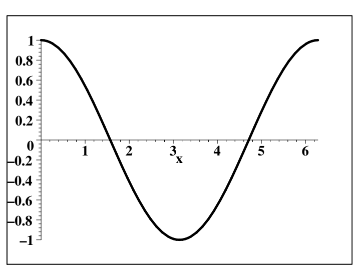
1. In this class (and future math classes) we will always use radians, never degrees.

2. Graphs of the basic trigonometric functions on $[0, 2\pi]$:

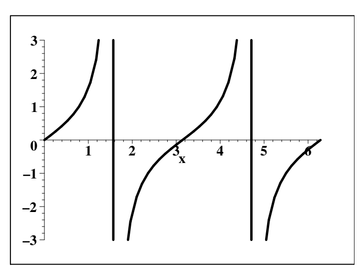
$f(x) = \sin x$



$f(x) = \cos x$



$f(x) = \tan x$



3. In the first quadrant, the sine, cosine, and tangent functions have the following values:

x	$\sin x$	$\cos x$	$\tan x$
0	0	1	0
$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/2$	1	0	—

   You should also be able to figure out the corresponding values of these functions in the other three quadrants.

4. There are three other trigonometric functions: secant, cosecant, and cotangent. These are the reciprocals of cosine, sine, and tangent respectively. That is:

$$\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x} \qquad \cot x = \frac{1}{\tan x}$$

5. Pythagorean identities and even-odd identities:

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 \\ 1 + \tan^2 x &= \sec^2 x \\ 1 + \cot^2 x &= \csc^2 x \end{aligned} \qquad \begin{aligned} \sin(-x) &= -\sin x \\ \cos(-x) &= \cos x \\ \tan(-x) &= -\tan x \end{aligned}$$

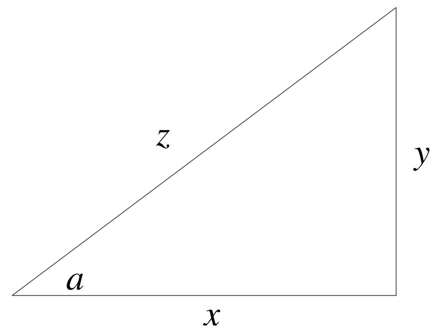
6. The **half-angle formulas** for sine and cosine let you rewrite $\sin^2 x$ and $\cos^2 x$. The **double-angle formulas** for sine and cosine let you rewrite $\sin(2x)$ and $\cos(2x)$. The formulas can be found at the beginning of your textbook. Although you do not need to have these formulas memorized, you do need to know that they exist, so that if one of the above expressions appears in a homework question, you should immediately think of these formulas. If one of these formulas is required on an exam, it will be given to you.

7. Simlarly, you should know that the sine and cosine addition formulas exist, and they allow you to rewrite $\sin(x + y)$ and $\cos(x + y)$.

8. The expression $(\sin x)^2$ is usually written as $\sin^2 x$ out of laziness. These two expressions are *not equal* to $\sin(x^2)$.

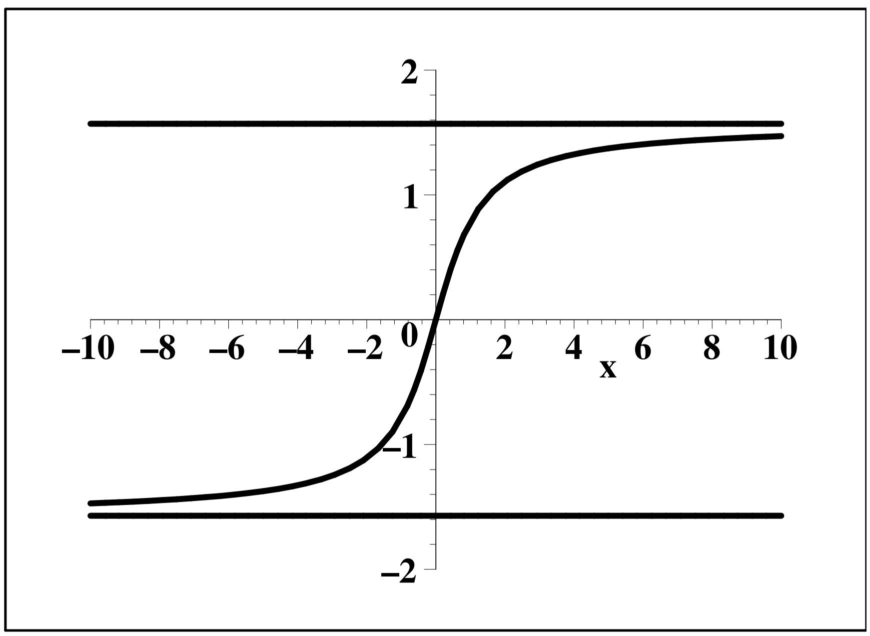
9. Given the right triangle below with angle a, we have the formulas

$$\sin a = \frac{y}{z} \qquad \cos a = \frac{x}{z} \qquad \tan a = \frac{y}{x}$$



10. The inverse tangent function, $y = \arctan x$, has horizontal asymptotes of $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$.

$f(x) = \arctan x$



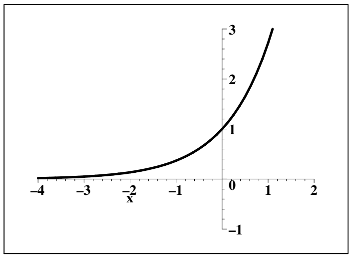
5 Exponentials and Logarithms

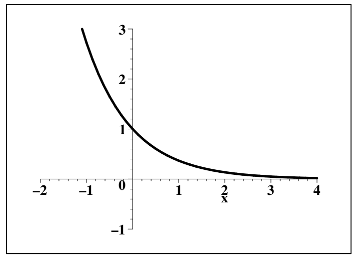
1. An **exponential function** has the form $y = a^x$ for some positive number a. While a is often a positive integer, the most common value for a is the number e, which is approximately 2.718. At some point during this class, we will talk more about e, but for now, you should simply consider it a number, like π. We will often talk about the function $y = e^x$.

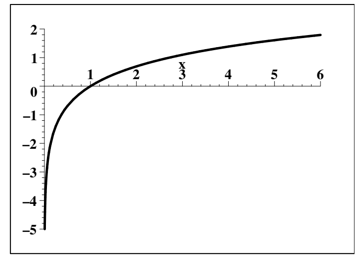
2. A **logarithm function** has the form $y = \log_b x$ for some positive number b. While b is sometimes 2 or 10, the most common value for b is e. In this case, we have the **natural logarithm function**, and it can be written either as $y = \log_e x$ or as $y = \ln x$. We will learn in this class why a and b are often chosen to be the number e.

3. Graphs of exponential and logarithm functions:

$f(x) = e^x$ $\qquad$ $f(x) = e^{-x}$ $\qquad$ $f(x) = \ln x$







4. There are certain special values of e^x and $\ln x$ that you should know. For e^x, you should know that $e^0 = 1$ and $e^1 = e$. For $\ln x$, you should know that $\ln 1 = 0$ and $\ln e = 1$.

5. Logarithms obey three very useful rules:

$$\ln ab = \ln a + \ln b \qquad \ln \frac{a}{b} = \ln a - \ln b \qquad \ln\left(a^b\right) = b \ln a$$

It is easy to be confused about what is and is not allowed. Here are some examples of what is not allowed:

$$\ln a + b \neq \ln a + \ln b \qquad \ln a + b \neq \ln a \ln b \qquad (\ln a)^b \neq b \ln a$$

6. Exponential functions and logarithm functions are related in a very important way. In particular, e^x and $\ln x$ are called **inverse functions** because they "cancel each other out." That is, $e^{\ln x} = x$ and $\ln e^x = x$. So "doing one and then the other" (e.g. raising e to the power of x, and then taking the natural logarithm of what you get) gives you what you started with.

7. If we ever have an exponential function a^x or a logarithm function $\log_b x$, and we would prefer to be using the number e, we can rewrite the original function as follows:

$$a^x = e^{a \ln x} \qquad \log_b x = \frac{\ln x}{\ln b}$$

6 More Geometry

1. The equation of a circle with center (a, b) and radius r is $(x-a)^2 + (y-b)^2 = r^2$. In particular, the equation of a circle centered at the origin with radius r is $x^2 + y^2 = r^2$. Solving this equation for y shows that $y = \pm\sqrt{r^2 - x^2}$. This shows that the equation of the upper semicircle centered at the origin with radius r is $y = \sqrt{r^2 - x^2}$. The equation of the lower semicircle is $y = -\sqrt{r^2 - x^2}$.

2. The area of a circle with radius r is $A = \pi r^2$. The circumference of a circle with radius r is $C = 2\pi r$.

3. The area of a triangle with base length b and height h is $A = \frac{1}{2}bh$.

4. The volume of a cylinder with radius r and height h is $V = \pi r^2 h$. The surface area of the side of the cylinder is $A = 2\pi rh$. The total surface area of the cylinder, including the top and bottom, is $A = 2\pi rh + 2\pi r^2$.

5. The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$. The surface area of a sphere of radius r is $A = 4\pi r^2$.

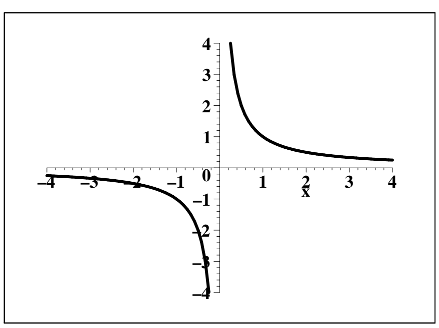
6. The volume of a cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$.

7 Other Topics

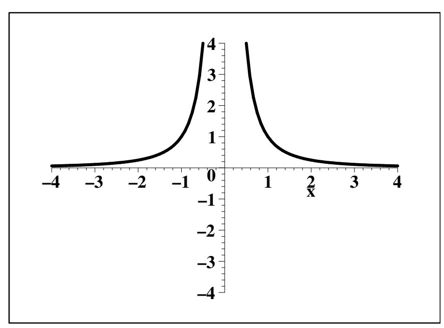
1. Throughout the course, we will have equations that have **constants** in them. For example, I might say: let $y = cx$, where c is a constant. This means that c is some fixed number, but we don't know what it is. In this case, we know that the graph of $y = cx$ is a straight line going through the point $(0, 0)$ with slope c – we just don't know what that slope c is. We often state rules using constants. For example, the Quadratic Formula says that if $0 = ax^2 + bx + c$, where $a \neq 0$, b, and c are constants, then
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$
Moreover, the Quadratic Formula is still true if a, b, and c are replaced by any numbers. However, the Quadratic Formula is not still true if a is replaced by x, because x is not a constant. That is to say, if $0 = x \cdot x^2 + bx + c$, the Quadratic Formula is not the correct way to solve for x.

2. You should also know the following graphs:

$f(x) = \frac{1}{x}$



$f(x) = \frac{1}{x^2}$



3. There are two ways to write down intervals on the real line. Here are some examples (note that parentheses correspond to $<$ and $>$ and square brackets correspond to $\leq$

and $\geq$).

$x < 0$	$(-\infty, 0)$	$0 < x < 1$	$(0, 1)$
$x \leq 0$	$(-\infty, 0]$	$0 \leq x < 1$	$[0, 1)$
$0 < x$	$(0, \infty)$	$0 \leq x \leq 1$	$[0, 1]$
$0 \leq x$	$[0, \infty)$	all x	$(-\infty, \infty)$

If an answer consists of multiple intervals, they can be combined with a $\cup$ symbol as follows:

$x < 0$ or $x > 1$	$(-\infty, 0) \cup (1, \infty)$
$0 \leq x \leq 1$ or $2 \leq x$	$[0, 1] \cup [2, \infty)$

4. A **piecewise function** is a function given by different formulas on different intervals. For example,

$$f(x) = \begin{cases} x^2, & x < 0 \\ x, & 0 \leq x < \pi \\ \sin x, & x \geq \pi \end{cases}$$

is a piecewise function that is equal to x^2 when x is negative, equal to x when x is in the interval $[0, \pi)$, and equal to $\sin x$ when $x \geq \pi$. The graph of this function is: