Math 19: Calculus **Summer 2010**

Final Exam — August 13, 2010, 7:00 to 10:00 PM

Name: Solution Key

Section (circle one): Eric · 1:15 PM Anca · 2:15 PM

- **You have a maximum of 3 hours.** This is a closed-book, closed-notes exam. No calculators or other electronic aids are allowed.
- Read each question carefully. Show your work and justify your answers for full credit. You do not need to simplify your answers unless instructed to do so. You may use results from class, but if you cite a theorem you should check that the hypotheses are explicitly verified.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please sign to indicate that you have read and agree to the following statement:

  "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the Stanford Honor Code with respect to this examination."

Signature: ______________________

Grading

1	/10	**6**	/10
2	/20	**7**	/10
3	/20	**8**	/20
4	/10	**9**	/10
5	/10	**10**	/10
		Total	/130

1. (*10 points*) Circle "True" or "False." No explanation is needed.

(a) **True** $\boxed{\textbf{False}}$ $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \lim_{x\to a} \dfrac{f'(x)}{g'(x)}$.

(b) $\boxed{\textbf{True}}$ **False** $\dfrac{d}{dx}(\arccos x) = -\dfrac{1}{\sqrt{1-x^2}}$.

(c) $\boxed{\textbf{True}}$ **False** $f(x) = |x+1|$ is differentiable at $x = 1$.

(d) **True** $\boxed{\textbf{False}}$ If $f'(2) = 0$ and $f''(2) = 4$, then f has a local maximum at $x = 2$.

(e) **True** $\boxed{\textbf{False}}$ An equation of the tangent line to the graph $y = \sin x$ at $(\pi, 0)$ is $y = \cos x(x - \pi)$.

(f) $\boxed{\textbf{True}}$ **False** If f is one-to-one and differentiable, then $\dfrac{d}{dx}[f^{-1}(x)] = \dfrac{1}{f'(f^{-1}(x))}$.

(g) **True** $\boxed{\textbf{False}}$ If $f''(2) = 0$, then 2 is an inflection point of f.

(h) $\boxed{\textbf{True}}$ **False** If f and g are increasing on an interval, $f + g$ is increasing on that interval.

(i) $\boxed{\textbf{True}}$ **False** If f is differentiable, then $(f(\sqrt{x}))' = \dfrac{f'(\sqrt{x})}{2\sqrt{x}}$.

(j) **True** $\boxed{\textbf{False}}$ If $f'(c) = 0$, then f has either a local maximum or a local minimum at c.

2. (*20 points*) Compute the derivatives of the following functions.

(a) (*5 points*) $f(x) = x \arcsin(x) + \sqrt{1-x^2}$ (Simplify as much as possible.)

Solution:

$$f'(x) = \arcsin(x) + \frac{x}{\sqrt{1-x^2}} + \frac{-2x}{\sqrt{1-x^2}} = \boxed{\arcsin(x).}$$

(b) (*5 points*) $g(u) = (\cos u)^u$

Solution: $\ln g(u) = \ln((\cos u)^u) = u \ln(\cos u)$, so

$$\begin{aligned} \frac{g'(u)}{g(u)} &= \ln(\cos u) + u\frac{-\sin u}{\cos u} = \ln(\cos u) - u \tan u \\ g'(u) &= \boxed{(\cos u)^u(\ln(\cos u) - u\tan u).} \end{aligned}$$

2., continued.

(c) (*5 points*) $h(x) = \ln\left(\ln\left(\dfrac{x}{x^2+1}\right)\right)$

Solution:

$$\begin{aligned} h'(x) &= \frac{1}{\ln\left(\dfrac{x}{x^2+1}\right)} \cdot \frac{x^2+1}{x} \cdot \frac{x^2+1-(x)(2x)}{(x^2+1)} \\ &= \frac{(x^2+1)(1-x^2)}{\ln\left(\dfrac{x}{x^2+1}\right) x(x^2+1)^2} \\ &= \boxed{\frac{1-x^2}{\ln\left(\dfrac{x}{x^2+1}\right) x(x^2+1)}.} \end{aligned}$$

(d) (*5 points*) $k(z) = \dfrac{e^z + e^{-z}}{e^z - e^{-z}}$

Solution:

$$\begin{aligned} k'(z) &= \frac{(e^z - e^{-z})(e^z - e^{-z}) - (e^z + e^{-z})(e^z + e^{-z})}{(e^z - e^{-z})^2} \\ &= \frac{e^{2z} - 2 + e^{-2z} - e^{2z} - 2 - e^{-2z}}{(e^z - e^{-z})^2} \\ &= \boxed{\frac{-4}{(e^z - e^{-z})^2}.} \end{aligned}$$

3. (*20 points*) Compute the following limits. Explain your reasoning where appropriate.

(a) (*5 points*) $\lim_{x\to 2^+} e^{3/(2-x)}$

Solution:

$$\lim_{x\to 2^+} e^{3/(2-x)} = e^{3/0^-} = e^{-\infty} = \boxed{0.}$$

(b) (*5 points*) $\lim_{x\to\infty} xe^{-x^2}$

Solution:

$$\begin{aligned}\lim_{x\to\infty} xe^{-x^2} &= \lim_{x\to\infty} \frac{x}{e^{x^2}} = \frac{\infty}{\infty} \\ &\overset{\text{LH}}{=} \lim_{x\to\infty} \frac{1}{2xe^{x^2}} = \frac{1}{\infty} = \boxed{0.}\end{aligned}$$

3., continued.

(c) (*5 points*) $\lim_{x\to 1} \dfrac{1-x+\ln x}{1+\cos \pi x}$

Solution:

$$\begin{aligned}
\lim_{x\to 1} \frac{1-x+\ln x}{1+\cos \pi x} &= \frac{1-1+0}{1+\cos \pi} = \frac{0}{0} \\
&\overset{\text{LH}}{=} \lim_{x\to 1} \frac{\frac{1}{x}-1}{-\pi \sin \pi x} = \frac{0}{0} \\
&\overset{\text{LH}}{=} \lim_{x\to 1} \frac{-\frac{1}{x^2}}{-\pi^2 \cos \pi x} = \frac{-1}{-\pi^2(-1)} = \boxed{\frac{-1}{\pi^2}.}
\end{aligned}$$

(d) (*5 points*) $\lim_{x\to \infty} \dfrac{(\ln x)^2}{x}$

Solution:

$$\begin{aligned}
\lim_{x\to\infty} \frac{(\ln x)^2}{x} &= \frac{\infty}{\infty} \\
&\overset{\text{LH}}{=} \lim_{x\to\infty} \frac{2\ln x \frac{1}{x}}{1} = \lim_{x\to\infty} \frac{2\ln x}{x} = \frac{\infty}{\infty} \\
&\overset{\text{LH}}{=} \lim_{x\to\infty} \frac{\frac{2}{x}}{1} = \frac{2}{\infty} = \boxed{0.}
\end{aligned}$$

4. (*10 points*) For each part, explain your work.

(a) (*7 points*) Use a linear approximation to estimate $(3.001)^4$.

Solution: Let $f(x) = x^4$ and let $a = 3$ (close to 3.001). Then

$$f(x) \approx L(x) = f(a) + f'(a)(x-a).$$

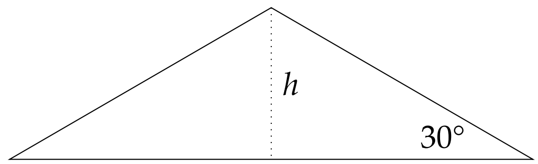
Since $f'(x) = 4x^3$, $f(a) = 3^4 = 81$, and $f'(a) = 4(3^3) = 108$. Hence,

$$f(3.001) \approx 81 + 108(3.001 - 3) = 81 + 108(0.001) = \boxed{81.108.}$$

(b) (*3 points*) Is your answer to part (a) an overestimate or an underestimate?

Solution: We compute that $f''(x) = 12x^2$, so $f''(3) = 12(3^2)$, which is positive. Therefore, f is concave up near $x = 3$, so the tangent line lies below the curve, and the approximation is therefore an $\boxed{\text{underestimate.}}$

5. (*10 points*) Sand pours off of a conveyer belt into a cone-shaped pile at a rate of 24 m^3/min. The angle that the side of the cone makes with the ground, known as the *angle of repose*, is $\frac{\pi}{6}$ radians (i.e., 30°). When the height of the cone is 4 m, how fast is the height of the cone changing?



Solution: Let r denote the radius of the base of the cone. From the diagram, $\frac{h}{r} = \tan 30^\circ = \frac{1}{\sqrt{3}}$, so $r = h\sqrt{3}$. Hence, the volume of the cone is

$$V = \frac{\pi}{3} r^2 h = \frac{\pi}{3}(\sqrt{3}h)^2 h = \pi h^3.$$

Differentiating,

$$\frac{dV}{dt} = 3\pi h^2 \frac{dh}{dt}.$$

Since $\frac{dV}{dt} = 24 \text{ m}^3/\text{min}$ and $h = 4$ m,

$$\frac{dh}{dt} = \frac{24 \frac{\text{m}^3}{\text{min}}}{3\pi(4 \text{ m})^2} = \boxed{\frac{1}{2\pi} \frac{\text{m}}{\text{min}}.}$$

6. (*10 points*) Let $g(t) = t^{4/3} - 4t^{1/3}$, on $[-1, 8]$.

(a) (*5 points*) Find the critical points (numbers) of g.

Solution: We differentiate $g(t)$:

$$g'(t) = \frac{4}{3}t^{1/3} - \frac{4}{3}t^{-2/3}$$

For the critical numbers, we check where $g'(t)$ is not defined and where $g'(t) = 0$. Because of the $t^{-2/3}$ term, $g'(t)$ is not defined at $t = 0$, but it is defined at all other t. Setting $g'(t) = 0$, we have

$$\frac{4}{3}t^{1/3} - \frac{4}{3}t^{-2/3} = 0 \quad \Rightarrow \quad \frac{4}{3}t^{1/3} = \frac{4}{3}t^{-2/3}.$$

Dividing both sides by $\frac{4}{3}t^{-2/3}$, we have $t = 1$. Hence, the critical points occur at $\boxed{t = 0 \text{ and } t = 1.}$

(b) (*5 points*) Find the absolute maximum and absolute minimum of g on the given interval.

Solution: We check the values of g at the critical points $t = 0$ and $t = 1$, as well as at the endpoints $t = -1$ and $t = 8$:

$$g(0) = 0 - 4(0) = 0, \qquad g(1) = 1^{4/3} - 41^{1/3} = 1 - 4 = -3,$$
$$g(-1) = (-1)^{4/3} - 4(-1)^{1/3} = 1 + 4 = 5, \quad g(8) = (8)^{4/3} - 4(8)^{1/3} = 16 - 8 = 8.$$

Hence, the absolute maximum occurs at $\boxed{x = 8,}$ and the absolute minimum at $\boxed{x = 1.}$

7. (*10 points*) Find values of a and b that make the function

$$f(x) = \begin{cases} ae^x(x^2-1) + 5b\cos x, & x \le 0, \\ (a+b)x\ln(e^3x), & 0 < x \le 1, \\ a(x+1)^2 - \frac{8}{\pi}\arctan(x), & x \ge 1 \end{cases}$$

continuous for all real x.

Solution: We note that these functions are defined and continuous on the specified domains, so we need only check continuity at the transition points $x = 0$ and $x = 1$. At $x = 0$, the limit from the left is

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} ae^x(x^2-1) + 5b\cos x = ae^0(0^2-1) + 5b\cos(0) = 5b - a.$$

The limit from the right is

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (a+b)x\ln(e^3x).$$

Direct substitution of $x = 0$ gives $(a+b)(0)(-\infty)$, so we need to rewrite the limit. Writing the limit as

$$\lim_{x\to 0^+} (a+b)x\ln(e^3x) = \lim_{x\to 0^+} (a+b)\frac{\ln(e^3x)}{\frac{1}{x}},$$

direct substitution gives $\frac{-\infty}{\infty}$, so we can apply l'Hôpital's rule. Doing so gives

$$\lim_{x\to 0^+} (a+b)\frac{\ln(e^3x)}{\frac{1}{x}} = \lim_{x\to 0^+} (a+b)\frac{\frac{e^3}{e^3x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} (a+b)(-x) = 0.$$

Hence, for this function to be continuous at 0, we need $5b - a = 0$, or $a = 5b$.
Next, we check for continuity at $x = 1$. Here,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (a+b)x\ln(e^3x) = (a+b)\ln e^3 = 3(a+b),$$
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} a(x+1)^2 - \frac{8}{\pi}\arctan(x) = a(2)^2 - \frac{8}{\pi}\frac{\pi}{4} = 4a - 2.$$

Thus, $3a + 3b = 4a - 2$, so $a = 3b + 2$. Eliminating a using the previous equation, we have $5b = 3b + 2$, so $2b = 2$, and $\boxed{b = 1.}$ Hence, $\boxed{a = 5.}$

8. (*20 points*) Let $f(x) = \dfrac{x^2e^{-x}}{x+1}$.

(a) (*2 points*) Find the domain of f and the x-values for which $f(x) = 0$.

Solution: Since x^2e^{-x} and $x+1$ are defined for all real x, $f(x)$ is defined where the denominator $x+1$ is not equal to 0. Thus, the domain is $\boxed{x \neq -1.}$

$f(x)$ is equal to 0 when the numerator $x^2e^{-x} = 0$. Since $e^{-x} \neq 0$, $x^2 = 0$, so $\boxed{x = 0}$ is the only zero of $f(x)$.

(b) (*4 points*) Find the vertical and horizontal asymptotes of $f(x)$, if any. Describe the behavior of f at each vertical asymptote.

Solution: For the vertical asymptotes, we check for infinite discontinuities of f. Since the numerator and denominator are both continuous for all real x, a discontinuity will occur only when the denominator is 0, so at $x = -1$. Since the numerator is $(-1)^2e^{-(-1)} = e^1 = e$, which is nonzero, there is an infinite discontinuity, and hence a vertical asymptote, at $\boxed{x = -1.}$ We also check the behavior on either side:

$$\lim_{x\to -1^-} \frac{x^2e^{-x}}{x+1} = \frac{e}{0^-} = -\infty \qquad \lim_{x\to -1^-} \frac{x^2e^{-x}}{x+1} = \frac{e}{0^+} = +\infty$$

For the horizontal asymptotes, we check the limits at $\pm\infty$:

$$\lim_{x\to-\infty} \frac{x^2}{x+1}e^{-x} = \lim_{x\to-\infty} \frac{x}{1+\frac{1}{x}}e^{-x} = \left(\frac{-\infty}{1}\right)(e^{\infty}) = -\infty.$$

The limit as $x \to \infty$ yields $\frac{\infty \cdot 0}{\infty}$ on direct substitution, so we rewrite it to apply l'Hôpital's rule:

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2e^{-x}}{x+1}e^{-x} &= \lim_{x\to\infty} \frac{x^2}{(x+1)e^x} = \frac{\infty}{\infty} \\
&\overset{LH}{=} \lim_{x\to\infty} \frac{2x}{(x+2)e^x} = \frac{\infty}{\infty} \\
&\overset{LH}{=} \lim_{x\to\infty} \frac{2}{(x+3)e^x} = \frac{2}{\infty} = 0.
\end{aligned}$$

Hence, $\boxed{y = 0}$ is the only horizontal asymptote, and $f(x)$ decreases without bound as $x \to -\infty$.

8., continued.

(c) (*4 points*) Find $f'(x)$ and simplify as much as possible. Find the critical numbers of f.

Solution: We compute $f'(x)$:

$$\begin{aligned} f'(x) &= \frac{(x+1)(2xe^{-x} - x^2e^{-x}) - x^2e^{-x}}{(x+1)^2} \\ &= \frac{xe^{-x}[(x+1)(2-x) - x]}{(x+1)^2} = \frac{x(2-x^2)e^{-x}}{(x+1)^2}. \end{aligned}$$

The only x-value for which $f'(x)$ does not exist is $x = -1$, but that is not in the domain of $f(x)$. Next, $f'(x) = 0$ when the numerator $x(2-x^2)e^{-x} = 0$. Since $e^{-x} \neq 0$, this occurs when $\boxed{x = 0,\ x = \sqrt{2},\ \text{and } x = -\sqrt{2}.}$

(d) (*3 points*) Find the intervals of increase and decrease.

Solution: We check the sign of $f'(x)$ on the intervals between the critical points $x = 0$, $x = \pm\sqrt{2}$ and the discontinuity $x = -1$. Since $e^{-x} > 0$ and since $(x+1)^2 \geq 0$, the sign of $f'(x)$ is equal to the sign of the remaining factors, $x(2-x^2)$.

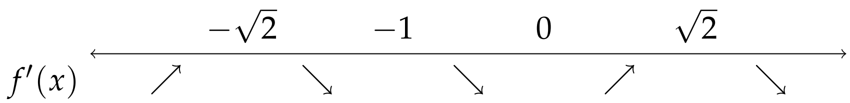
- For $x < -\sqrt{2}$, $x < 0$ and $2 - x^2 < 0$, so $f'(x) > 0$.
- For $-\sqrt{2} < x < -1$, $x < 0$ and $2 - x^2 > 0$, so $f'(x) < 0$.
- For $-1 < x < 0$, $x < 0$ and $2 - x^2 > 0$, so $f'(x) < 0$.
- For $0 < x < \sqrt{2}$, $x > 0$ and $2 - x^2 > 0$, so $f'(x) > 0$.
- For $x > \sqrt{2}$, $x > 0$ and $2 - x^2 < 0$, so $f'(x) < 0$.

Therefore, f is increasing on $\boxed{(-\infty, -\sqrt{2}) \text{ and on } (0, \sqrt{2}),}$ and f is decreasing on $\boxed{(-\sqrt{2}, -1),\ (-1, 0),\ \text{and } (\sqrt{2}, \infty).}$

8., continued.

(e) (*3 points*) Find all local minimum or maximum values and where they occur.

Solution: Since we computed the intervals of increase and decrease in part (d), we use the First Derivative Test. We have the following depiction of the number line:

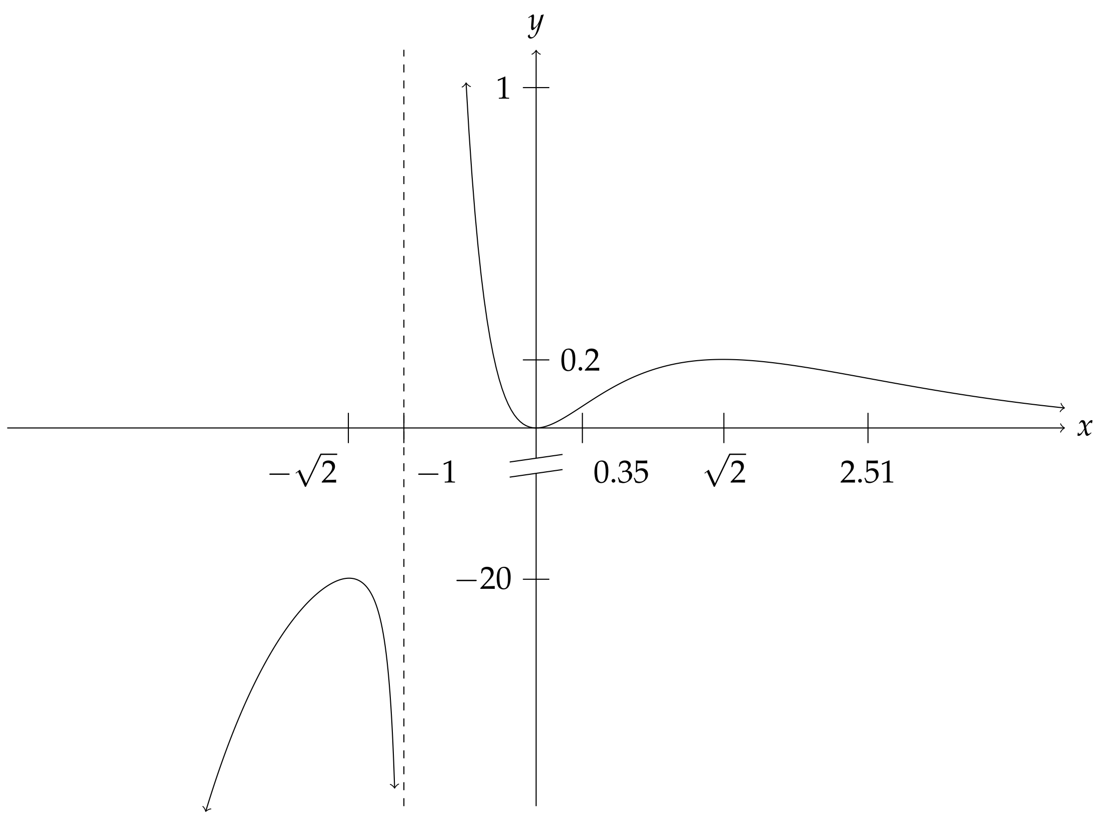


Therefore, f has local maxima at $x = \sqrt{2}$ and $x = -\sqrt{2}$, and a local minimum at $x = 0$. The y-values are

$$f(-\sqrt{2}) = \frac{(\sqrt{2})^2 e^{-\sqrt{2}}}{1-\sqrt{2}} = \frac{2e^{-\sqrt{2}}}{1-\sqrt{2}}, \qquad f(0) = 0, \qquad f(\sqrt{2}) = \frac{2e^{\sqrt{2}}}{1+\sqrt{2}}.$$

These values are approximately -20, 0, and 0.2 (although we did not expect you to compute these on the exam).

(f) (*4 points*) f is concave down on the intervals $(-\infty, -1)$ and $(0.35, 2.51)$, and is concave up on the intervals $(-1, 0.35)$ and $(2.51, +\infty)$. Use this and the information above to sketch the graph of f. Make sure to mark any key x- and y-values on the axes.



9. (*10 points*) Consider the following equation: $y\sin(2x) = x\cos(2y)$.

(a) (*6 points*) Use implicit differentiation to find $\dfrac{dy}{dx}$.

Solution:

$$\begin{aligned}
\frac{dy}{dx}\sin(2x) + 2y\cos(2x) &= \cos(2y) - 2x\sin(2y)\frac{dy}{dx} \\
\frac{dy}{dx}\sin(2x) + 2x\sin(2y)\frac{dy}{dx} &= \cos(2y) - 2y\cos(2x) \\
\frac{dy}{dx}(\sin(2x) + 2x\sin(2y)) &= \cos(2y) - 2y\cos(2x) \\
\frac{dy}{dx} &= \boxed{\frac{\cos(2y) - 2y\cos(2x)}{\sin(2x) + 2x\sin(2y)}}
\end{aligned}$$

(b) (*4 points*) Find the equation of the tangent line to the curve at the point $\left(\dfrac{\pi}{2}, \dfrac{\pi}{4}\right)$.

Solution: At $(x, y) = (\frac{\pi}{2}, \frac{\pi}{4})$,

$$\frac{dy}{dx} = \frac{\cos\frac{\pi}{2} - 2\frac{\pi}{4}\cos\pi}{\sin\pi + 2\frac{\pi}{2}\sin\frac{\pi}{2}} = \frac{\pi/2}{\pi} = \frac{1}{2}.$$

Therefore, the tangent line is

$$y - \frac{\pi}{2} = \frac{1}{2}\left(x - \frac{\pi}{2}\right) \quad \Rightarrow \quad \boxed{y = \frac{1}{2}x.}$$

10. (*10 points*)

(a) (*3 points*) If $C(x)$ is the cost of producing x units of a commodity, then the average cost per unit is $a(x) = \dfrac{C(x)}{x}$. Show that if the average cost is a minimum, then the marginal cost $C'(x)$ equals the average cost. (Assume $C(x)$ is differentiable.)

Solution: At a minimum, $a'(x) = 0$. Then

$$0 = \left(\frac{C(x)}{x}\right)' = \frac{xC'(x) - C(x)}{x^2},$$

so $xC'(x) - C(x) = 0$, and $C'(x) = \frac{C(x)}{x} = a(x)$.

(b) (*7 points*) If $C(x) = 16{,}000 + 200x + 4x^{3/2}$, find the minimum average cost and the production level at that minimum.

Solution: The average cost is

$$a(x) = \frac{16{,}000}{x} + 200 + 4x^{1/2},$$

so

$$a'(x) = -\frac{16{,}000}{x^2} + 2x^{-1/2} = \frac{2x^{3/2} - 16{,}000}{x^2}.$$

When $a'(x) = 0$, $2x^{3/2} - 16{,}000$, so $x^{3/2} = 8000$, and $x = 400$. Hence, this is the only critical point of the average cost.
Note that $a(x)$ is defined on $(0, \infty)$. Since

$$\lim_{x \to 0} a(x) = +\infty \quad \text{and} \quad \lim_{x \to \infty} a(x) = +\infty,$$

$a(x)$ must have an absolute minimum at $\boxed{x = 400.}$ Finally,

$$a(400) = \frac{16{,}000}{400} + 200 + 4(200)^{1/2} = 40 + 200 + 80 = \boxed{320}$$

is the minimum average cost.