Practice Final Exam: Summer 2009 – Solutions

1. Circle "True" or "False." No explanation is needed.

(a) **[True]** **False** If $f'(x) < 0$ for $1 < x < 6$, then $f(x)$ is decreasing on $(1, 6)$.
Solution: $f'(x) < 0$ means $f(x)$ is decreasing.

(b) **True** **[False]** If $f(x)$ has an local minimum value at $x = c$, then $f'(c) = 0$.
Solution: It might be the case that $f'(c)$ does not exist (e.g., $f(x) = |x|$ at $c = 0$).

(c) **True** **[False]** $f'(x)$ has the same domain as $f(x)$.
Solution: The domain of $f'(x)$ might be strictly smaller (e.g., $f(x) = |x|$).

(d) **True** **[False]** If both $f(x)$ and $g(x)$ are differentiable, then $\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}f(x) \cdot \frac{d}{dx}g(x)$.
Solution: The product rule states $\frac{d}{dx}(f(x)g(x)) = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x)$.

(e) **True** **[False]** A function has at most two vertical asymptotes.
Solution: $f(x) = \tan x$ has infinitely many vertical asymptotes.

2. For each of the following limits, find its value or explain why it does not exist. If the limit involves infinity, explain whether it is ∞ or $-\infty$.

(a) $\lim_{x \to 0^+} \frac{\cos x}{x}$
Solution:

$$\lim_{x \to 0^+} \frac{\cos x}{x} = \frac{1}{0^+} = \boxed{+\infty.}$$

(b) $\lim_{x \to \infty} \frac{\sqrt{x^2 - 6}}{x + 6}$
Solution:

$$\lim_{x \to \infty} \frac{\sqrt{x^2 - 6}}{x + 6} = \lim_{x \to \infty} \frac{\sqrt{x^2 - 6}}{x + 6} \cdot \frac{1/x}{1/x} = \lim_{x \to \infty} \frac{\sqrt{1 - \frac{6}{x^2}}}{1 + \frac{6}{x}} = \frac{1}{1} = \boxed{1.}$$

(c) $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$

$$\begin{aligned}\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right) &= \lim_{x\to 0^+}\frac{\sin x - x}{x\sin x} = \frac{0}{0} \quad \text{(apply l'Hôpital)}\\ &= \lim_{x\to 0^+}\frac{\cos x - 1}{\sin x + x\cos x} = \frac{0}{0} \quad \text{(apply l'Hôpital)}\\ &= \lim_{x\to 0^+}\frac{-\sin x}{\cos x + \cos x - x\sin x} = -\frac{0}{1+1} = \boxed{0.}\end{aligned}$$

(d) $\lim_{x\to 0}(e^x + x)^{\frac{1}{x}}$

Solution: Since direct substitution gives 1^∞, we take logarithms and analyze $\ln(e^x + x)^{\frac{1}{x}} = \frac{1}{x}\ln(e^x + x)$:

$$\begin{aligned}\lim_{x\to 0}\frac{\ln(e^x + x)}{x} &= \frac{0}{0} \quad \text{(apply l'Hôpital)}\\ &= \lim_{x\to 0}\frac{\frac{e^x+1}{e^x+x}}{1} = \lim_{x\to 0}\frac{e^x+1}{e^x+x} = \frac{1+1}{1+0} = 2\end{aligned}$$

Exponentiating, the original limit is $\boxed{e^2.}$

3. For each function, find its derivative, using any method you prefer.

(a) $f(x) = \pi^{2x-7} + \sqrt{1-\sqrt{1-x^4}}$

Solution: Using the chain rule,

$$\begin{aligned}f'(x) &= 2\pi^{2x-7}\ln\pi + \frac{1}{2\sqrt{1-\sqrt{1-x^4}}}(-1)\frac{1}{2\sqrt{1-x^4}}(-4x^3)\\ &= \boxed{2\pi^{2x-7}\ln\pi + \frac{x^3}{\sqrt{1-\sqrt{1-x^4}}\sqrt{1-x^4}}.}\end{aligned}$$

(b) $g(x) = e^x(7x^2 + \arcsin x^2)$

Solution:

$$g'(x) = \boxed{e^x(7x^2 + \arcsin x^2) + e^x\left(14x + \frac{2x}{\sqrt{1-x^4}}\right)}$$

(c) $h(x) = \frac{(x^2-2)^3}{(x+3)^5\sqrt{x+1}}$

Solution:

$$\begin{aligned}
\ln h(x) &= \ln \frac{(x^2-2)^3}{(x+3)^5\sqrt{x+1}} = 3\ln(x^2-2) - 5\ln(x+3) - \frac{1}{2}\ln(x+1) \\
\frac{h'(x)}{h(x)} &= \frac{6x}{x^2-2} - \frac{5}{x+3} - \frac{1}{2(x+1)} \\
h'(x) &= h(x)\left(\frac{6x}{x^2-2} - \frac{5}{x+3} - \frac{1}{2(x+1)}\right) \\
&= \boxed{\frac{(x^2-2)^3}{(x+3)^5\sqrt{x+1}}\left(\frac{6x}{x^2-2} - \frac{5}{x+3} - \frac{1}{2(x+1)}\right)}
\end{aligned}$$

(d) $k(t) = \cos(t^{1/t})$

Solution: Let $j(t) = t^{1/t}$. Then $\ln j(t) = \frac{1}{t}\ln t$, so

$$\frac{j'(t)}{j(t)} = \frac{1}{t^2} - \frac{1}{t^2}\ln t = \frac{1-\ln t}{t^2} \quad \Rightarrow \quad j'(t) = t^{1/t}\frac{1-\ln t}{t^2}.$$

Hence,

$$k'(t) = \boxed{-\sin(t^{1/t})t^{1/t}\frac{1-\ln t}{t^2}.}$$

4. Answer the following questions.

(a) Complete the definition:

A function $f(x)$ is differentiable at $x = a$ if $\underline{\ f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \text{ exists}\ }$.

(b) Consider the function

$$f(x) = \begin{cases} x^3\sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Use the above definition to decide whether $f(x)$ is differentiable at $x = 0$.

Solution: We attempt to compute $f'(0)$:

$$f'(0) = \lim_{x\to 0}\frac{f(x+0)-f(0)}{x} = \lim_{x\to 0}\frac{x^3\sin(\frac{1}{x}) - 0}{x} = \lim_{x\to 0} x^2 \sin\frac{1}{x}$$

Since $-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2$ for all $x \neq 0$, and since $\lim_{x\to 0} x^2 = 0$, the Squeeze Theorem states that $\lim_{x\to 0} x^2\sin\frac{1}{x} = 0$ as well. Hence, $f'(0)$ exists, so f is differentiable at 0.

5. Answer the following questions.

(a) Give a precise statement of the Intermediate Value Theorem.

Solution: If $f(x)$ is a function continuous on the closed interval $[a,b]$ with $f(a) \neq f(b)$, then for each y between $f(a)$ and $f(b)$, there is some c in (a,b) with $f(c) = y$.

(b) Use the Intermediate Value Theorem to show that there exists a solution to the equation

$$\ln x = \sin\left(\frac{\pi}{2}x\right)$$

on the interval $(0, \infty)$.

Solution: Let $f(x) = \ln x - \sin\left(\frac{\pi}{2}x\right)$. At $x = e^{-2}$,

$$f(x) = -2 - \sin\left(\frac{\pi}{2}e^{-2}\right) < -2 + 1 = -1 < 0,$$

so $f(e^{-2}) < 0$. At $x = e^2$,

$$f(e^2) = 2 - \sin\left(\frac{\pi}{2}e^2\right) > 2 - 1 = 1 > 0,$$

so $f(e^2) > 0$. Hence, there is some c in (e^{-2}, e^2) such that $f(c) = 0$, and this c gives a solution to the original equation.

6. The equation $x^2y^2 + xy = 2$ describes a curve in the xy-plane.

(a) Find an expression for $\dfrac{dy}{dx}$.

Solution: Differentiating implicitly,

$$2xy^2 + 2x^2y\frac{dy}{dx} + y + x\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \boxed{-\frac{2xy^2 + y}{2x^2y + x}}.$$

(b) Find the equation of the line tangent to the curve at the point $(-1, 2)$.

Solution: At $(x,y) = (-1,2)$, $\frac{dy}{dx} = -\frac{2(-1)2^2+2}{2(-1)^2 2+(-1)} = \frac{6}{3} = 2$. Hence, the tangent line is $y - 2 = 2(x+1)$, or $\boxed{y = 2x + 4.}$

(c) Find the coordinates (x,y) of all points on the curve where the tangent line is parallel to the line $x + y = 1$.

(d) If the tangent line is parallel to $x + y = 1$, which has slope -1, then

$$-\frac{2xy^2 + y}{2x^2y + x} = -1 \quad \Rightarrow \quad 2xy^2 + y = 2x^2y + x \quad \Rightarrow \quad y(2xy+1) = x(2xy+1).$$

Thus, either $x = y$ or $xy = -\frac{1}{2}$. If $xy = -\frac{1}{2}$, then $y = -\frac{1}{2x}$, so, plugging this into the original equation $x^2y^2 + xy = 2$,

$$\frac{x^2}{4x^2} + \frac{x}{2x} = 2 \quad \Rightarrow \quad \frac{1}{4} + \frac{1}{2} = 2,$$

which is impossible. We conclude that $x = y$, so then $x^4 + x^2 - 2 = 0$. Thus, $(x^2 + 2)(x^2 - 1) = 0$; since $x^2 + 2 \neq 0$, $x^2 = 1$, so $x = 1$ or $x = -1$. Hence, the points are $\boxed{(1, 1) \text{ and } (-1, -1).}$

7. Consider the function $f(x) = x^{2/3}$.

(a) Find the linear approximation of the function $f(x)$ at the point $a = 8$; that is, find the linear function $L(x)$ that best approximates $f(x)$ for values of x near 8.

Solution: We compute $f'(x) = \frac{2}{3}x^{-1/3}$. At $a = 8$, $f(a) = 8^{2/3} = 4$ and $f'(a) = \frac{2}{3}8^{-1/3} = \frac{1}{3}$, so the linear approximation is

$$L(x) = \boxed{4 + \frac{1}{3}(x - 8).}$$

(b) Use the above linear approximation to estimate $(8.04)^{2/3}$. Is your approximation an overestimate or an underestimate of the actual value? Explain fully.

Solution: At $x = 8.04$, our linear approximation is

$$L(8.04) = 4 + \frac{1}{3}(8.04 - 8) = 4 + \frac{1}{3}(0.04) \approx \boxed{4.0133.}$$

Since $f''(x) = -\frac{2}{9}x^{-4/3}$, $f''(8)$ is negative, so $f(x)$ is concave down near $x = 8$. Hence, the graph of $y = f(x)$ lies below the tangent line at $x = 8$, so the value of $L(8.04)$ is higher than $f(8.04)$ and is therefore an overestimate.

8. Consider the function $f(x) = x^{1/3}(x - 8)^2$.

(a) Find all critical numbers of f.

Solution: We compute $f'(x)$:

$$\begin{aligned}
\ln f(x) &= \ln(x^{1/3}(x-8)^2) = \frac{1}{3}\ln x + 2\ln(x - 8) \\
\frac{f'(x)}{f(x)} &= \frac{1}{3x} + \frac{2}{x - 8} = \frac{x - 8 + 6x}{3x(x - 8)} \\
f'(x) &= x^{1/3}(x - 8)^2 \frac{7x - 8}{3x(x - 8)} = \frac{(7x - 8)(x - 8)}{3x^{2/3}}.
\end{aligned}$$

Then $f'(x)$ is undefined at $\boxed{x = 0,}$ and is 0 at $\boxed{x = 8 \text{ and } x = \frac{8}{7}.}$ Therefore, these are the critical numbers of f.

(b) Find the absolute maximum and minimum values of f on the interval $[-1, 8]$.

Solution: We check the value of f at the critical points above and at the endpoints $x = -1$ and $x = 8$:

$$
\begin{aligned}
f(0) &= 0^{1/3}(0-8)^2 = 0 \\
f(8) &= 8^{1/3}(8-8)^2 = 0 \\
f\left(\frac{8}{7}\right) &= \left(\frac{8}{7}\right)^{1/3}\left(\frac{8}{7}-8\right)^2 = \frac{2}{7^{1/3}}\left(-\frac{48}{7}\right)^2 = \frac{2(48)^2}{7^{7/3}} \\
f(-1) &= (-1)^{1/3}(-1-8)^2 = (-1)(-9)^2 = -81
\end{aligned}
$$

Since $f(\frac{8}{7})$ is the only positive value, it must be the absolute maximum, and since $f(-1)$ is the only negative value, it must be the absolute minimum.

9. Consider the function $f(x) = \dfrac{x^2}{x^2-1}$.

(a) Find the domain and zeroes of $f(x)$.

Solution: Note that f is defined where the denominator $x^2 - 1$ is not 0, and so is defined on $x \neq 1$ and $x \neq -1$. In interval notation, this is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

$f(x) = 0$ when the numerator is 0, so when $\boxed{x = 0.}$

(b) Find all horizontal and vertical asymptotes of $f(x)$. Justify your answer by limit computations.

Solution: We compute the horizontal asymptotes of f:

$$
\lim_{x\to\infty} \frac{x^2}{x^2-1} = \lim_{x\to\infty} \frac{x^2}{x^2-1}\frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{1}{1-\frac{1}{x^2}} = 1.
$$

Similarly, $\lim_{x\to-\infty} f(x) = 1$, so the only horizontal asymptote is $\boxed{y = 1.}$

We check for infinite discontinuities of f. Since f is a rational function, these occur when the denominator of f is 0 but the numerator of f is nonzero. The denominator is 0 when $x = 1$ and $x = -1$, and at these points the numerator x^2 is 1, so $\boxed{x = 1}$ and $\boxed{x = -1}$ are the two vertical asymptotes.

(c) Find $f'(x)$ and $f''(x)$, using any method you like.

Solution: We first write

$$
f(x) = \frac{x^2}{x^2-1} = \frac{x^2-1+1}{x^2-1} = 1 + \frac{1}{x^2-1}.
$$

Then $f'(x) = \boxed{-\dfrac{2x}{(x^2-1)^2}} = -2x(x^2-1)^{-2}$. Next, we differentiate $f'(x)$:

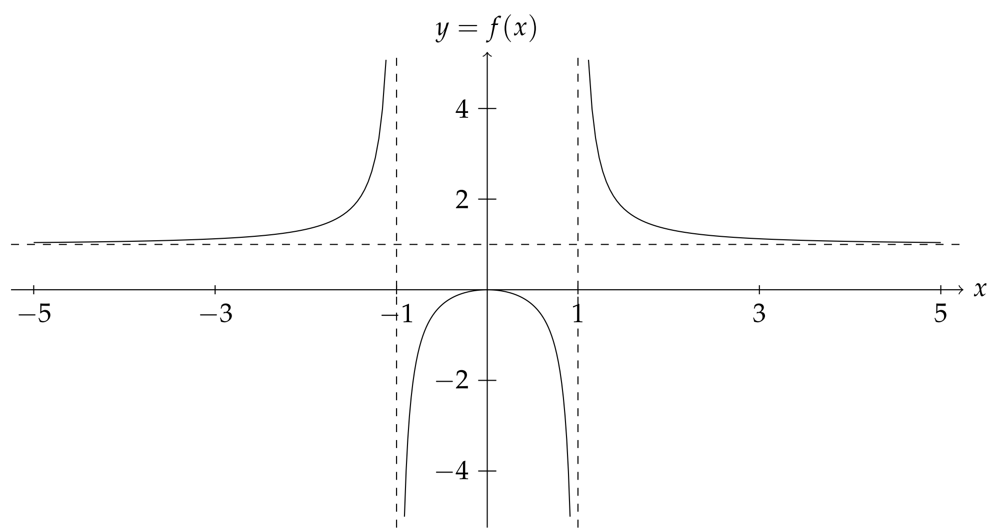
$$\begin{aligned} f''(x) &= -2((x^2-1)^{-2} + x(-2)(2x)(x^2-1)^{-3}) \\ &= \frac{-2}{(x^2-1)^3}(x^2-1-4x^2) = \boxed{\frac{2(3x^2+1)}{(x^2-1)^3}.} \end{aligned}$$

(d) Find the intervals of increase and decrease.
Solution: We check where $f'(x)$ is positive and negative. Since $f'(x) = \frac{-2x}{(x^2-1)^2}$ and since the denominator is positive, $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 0$. Then f is increasing on $\boxed{(-\infty, 0)}$ and decreasing on $\boxed{(0, \infty).}$

(e) Find all local maximum and local minimum values.
Solution: We first find the critical numbers of f: the only x-values where $f'(x)$ is undefined are $x = 1$ and $x = -1$, which are not in the domain of f. $f'(x) = 0$ when $x = 0$, so $x = 0$ is the only critical value. Since $f''(0) = \frac{2(1)}{(-1)^3} = -2$, f has a local maximum at $\boxed{x = 0.}$ Also, $f(0) = 0$.

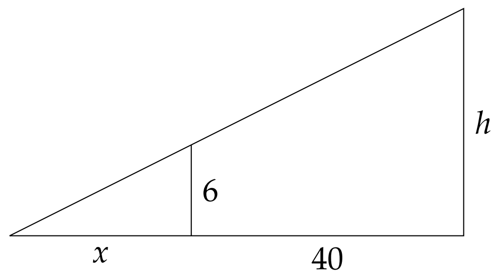
(f) Find the intervals of concavity and all inflection points.
Solution: We note that $2(3x^2+1) > 0$ for all x, so $f''(x)$ depends on the sign of the denominator $x^2 - 1$. For $|x| < 1$, $x^2 - 1 < 0$, so $f''(x) < 0$. For $|x| > 1$, $x^2 - 1 > 0$, so $f''(x) > 0$. Thus, f is concave up on $(-\infty, -1) \cup (1, \infty)$ and concave down on $(-1, 1)$. f has no inflection points, since it is undefined at the boundary points $x = 1$ and $x = -1$.

(g) Use the information from all above parts to sketch the graph of $f(x)$.
Solution:



10. Special Agent Fox Mulder, a 6-foot-tall man, notices a small UFO on the ground, located 40 feet from where he stands in a flat field. The UFO suddenly ascends vertically at a rate of 10 feet per second. Throughout the ascent, a bright light on the ship illuminates the entire field below, casting a shadow of Mulder onto the ground. What is the rate of change of the length of Mulder's shadow exactly three seconds after the UFO has taken off? (Hint: at any moment, the end of Mulder's shadow is always located on the ground, and on the line determined by the light source and Mulder's head.)

Solution: We produce a diagram of the situation, letting x denote the length of the shadow and letting h denote the height of the UFO:



From similar triangles, we have that

$$\frac{x}{6} = \frac{x+40}{h} \quad \Rightarrow \quad hx = 6x + 240 \quad \Rightarrow \quad x = \frac{240}{h-6}.$$

Therefore,

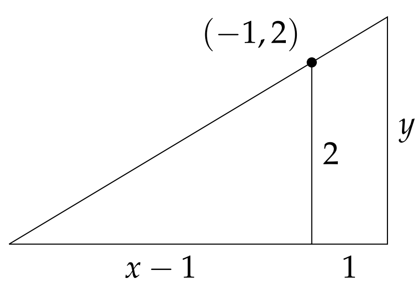
$$\frac{dx}{dt} = -\frac{240}{(h-6)^2}\frac{dh}{dt}.$$

Three seconds after takeoff, $h = 30$, so

$$\frac{dx}{dt} = -\frac{240}{(24)^2}10 = \boxed{-\frac{25}{6}\frac{\text{ft}}{\text{sec}}}.$$

11. In the xy-plane, any positively-sloped line that passes through the point $(-1, 2)$ will form a right triangle with the x-axis and y-axis in the second quadrant. Among all possible such lines (positive slope, passing through $(-1, 2)$), find the equation of the line that forms a triangle of minimal area. Justify completely.

Solution: We make the following diagram, where x represents the length of the base of the triangle and y the height:



Then the area of the triangle is $A = \frac{1}{2}xy$. By similar triangles, $\frac{x-1}{2} = \frac{x}{y}$, so $y = \frac{2x}{x-1}$, and

$$A = \frac{1}{2}xy = \frac{1}{2}x\frac{2x}{x-1} = \frac{x^2}{x-1}.$$

Geometrically, this area makes sense only for $x > 1$. Differentiating,

$$A'(x) = \frac{(x-1)(2x) - x^2}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2}.$$

$A'(x) = 0$ implies that $x^2 - 2x = 0$, so $x = 0$ or $x = 2$. Only $x = 2$ is in the domain of A, though, so $x = 2$ is the only critical point.

For $1 < x < 2$, $A'(x) < 0$, and for $x > 2$, $A'(x) > 0$, so A has an absolute minimum at $x = 2$.

At $x = 2$, $y = \frac{2^2}{2-1} = 4$, so the line has slope $\frac{4}{2} = 2$. Since it has a y-intercept of 4, the line is $\boxed{y = 2x + 4.}$