Homework #8 Solutions

Problems

- Section 3.5: 2, 6, 10, 20, 22, 26, 32, 34, 54, 58

3.5.2. Find a particular solution to the DE $y'' - y' - 2y = 3x + 4$.

Solution: We first prepare our guess for the particular function y_p. The 1 and x terms in $f(x) = 3x + 4$ correspond to a double root $r = 0$; examining the characteristic equation $r^2 - r - 2 = 0$, we immediately see that $r = 0$ is not a root, so there is no overlap with the complementary solution. Hence, we try

$$y_p = Ax + B.$$

Then $y_p' = A$ and $y_p'' = 0$, so plugging these into the DE we obtain $-A - 2(Ax + B) = 3x + 4$. From the x terms, $-2A = 3$, so $A = -3/2$. From the constant terms, $-A - 2B = 4$, so $B = \frac{1}{2}(-4 - A) = -\frac{5}{4}$. Thus, a particular solution is $y = -\frac{3}{2}x - \frac{5}{4}$. ■

3.5.6. Find a particular solution to the DE $2y'' + 4y' + 7y = x^2$.

Solution: In setting up the Method of Undetermined Coefficients, we observe that $f(x) = x^2$ corresponds to a triple root $r = 0$. Since the characteristic equation for the DE is $2r^2 + 4r + 7 = 0$, $r = 0$ is not one of its roots, so there is no overlap with x_c. Then we may take

$$y_p = Ax^2 + Bx + C$$

as a guess to plug into the DE. Doing so, and separating out the coefficients on the constant, x, and x^2 terms, we obtain the linear system

$$\begin{aligned} 7A &= 1 \\ 8A + 7B &= 0 \\ 4A + 4B + 7C &= 0 \end{aligned}$$

Solving for A, B, and C successively,

$$A = \frac{1}{7}, \quad B = -\frac{8}{7^2} = -\frac{8}{49}, \quad C = \frac{1}{7}\left(-\frac{4}{7} + \frac{32}{49}\right) = \frac{4}{343}$$

so therefore

$$y_p = \frac{1}{7}x^2 - \frac{8}{49}x + \frac{4}{343}$$

is one particular solution. ■

3.5.10. Find a particular solution to the DE $y'' + 9y = 2\cos 3x + 3\sin 3x$.

Solution: The roots corresponding to the $\cos 3x$ and $\sin 3x$ in $f(x)$ are $\pm 3i$, which are precisely the roots of the characteristic equation $r^2 + 9 = 0$ of the DE. Hence, to avoid overlap, we must guess a particular solution of the form

$$y_p = Ax\cos 3x + Bx\sin 3x.$$

Then $y_p' = A(\cos 3x - 3x\sin 3x) + B(\sin 3x + 3x\cos 3x)$, so

$$y_p'' = A(-6\sin 3x - 9x\cos 3x) + B(6\cos 3x - 9x\sin 3x),$$

and $y_p'' + 9y_p = -6A\sin 3x + 6B\cos 3x$. Since this is to equal $2\cos 3x + 3\sin 3x$, we see that $A = -3/6 = -1/2$, and $B = 2/6 = 1/3$, so a particular solution is

$$y = \frac{1}{3}x\sin 3x - \frac{1}{2}x\cos 3x.$$

■

3.5.20. Find a particular solution to the DE $y^{(3)} - y = e^x + 7$.

Solution: The roots corresponding to $f(x) = e^x + 7$ are $r = 0$ and $r = 1$. The characteristic equation is $r^3 - 1 = 0$, which has a root $r = 1$ as well. Factoring the polynomial as $(r-1)(r^2 + r + 1)$, we observe that the quadratic factor $r^2 + r + 1$ has no real roots, as its discriminant $b^2 - 4ac = 1 - 4 = -3$ is negative. Hence, the only overlap comes on the root $r = 1$, so our guess for a particular solution is

$$y_p = Axe^x + B.$$

We compute that $y_p^{(3)} = A(x+3)e^x$, so $y_p^{(3)} - y_p = 3Ae^x - B$. Since this is $e^x + 7$, $A = 1/3$ and $B = -7$, so a particular solution is

$$y = \frac{1}{3}xe^x - 7.$$

■

3.5.22. Set up the appropriate form for a particular solution to the DE $y^{(5)} - y^{(3)} = e^x + 2x^2 - 5$, but do not determine the values of the coefficients.

Solution: We note the roots coming from the chararcteristic equation and from the forcing term $f(x)$. The characteristic equation is $r^5 - r^3 = r^3(r-1)(r+1) = 0$, so its roots are $r = 1$, $r = -1$, and a triple root $r = 0$. On the forcing side, the e^x corresponds to $r = 1$, and the constant and x^2 terms correspond to another triple root $r = 0$. Thus, since $r = 1$ appears here for the second time total, and $r = 0$ appear for the fourth through sixth times, our guess is

$$y = Axe^x + Bx^3 + Cx^4 + Dx^5.$$

■

3.5.26. Set up the appropriate form for a particular solution to the DE $y'' - 6y' + 13y = xe^{3x}\sin 2x$, but do not determine the values of the coefficients.

Solution: The characteristic equation is $r^2 - 6r + 13 = 0$, which has roots

$$r = \frac{6 \pm \sqrt{36 - 4(13)}}{2} = 3 \pm 2i.$$

The $xe^{3x}\sin 2x$ term correponds to a double pair of roots $r = 3 \pm 2i$, which collide with the roots of the characteristic equation. Hence, our guess for a particular solution is of the form

$$y = Axe^{3x}\cos 2x + Bxe^{3x}\sin 2x + Cx^2e^{3x}\cos 2x + Dx^2e^{3x}\sin 2x.$$

■

3.5.32. Solve the initial value problem $y'' + 3y' + 2y = e^x$, $y(0) = 0$, $y'(0) = 3$.

Solution: We first determine the general complementary solution y_c and a particular solution y_p. Since the characteristic equation is $r^2 + 3r + 2 = 0$, $r = -1$ and $r = -2$ are its roots, and $y_c = c_1e^{-x} + c_2e^{-2x}$. Fortunately, neither root is $r = 1$, which corresponds to $f(x) = e^x$, so our guess for y_p is Ae^x. Plugging this into the DE,

$$A(1 + 3 + 2)e^x = e^x,$$

so $A = 1/6$. Hence, the general solution is

$$y = \frac{1}{6}e^x + c_1e^{-x} + c_2e^{-2x}, \qquad y' = \frac{1}{6}e^x - c_1e^{-x} - 2c_2e^{-2x}.$$

Applying the initial condition, $y(0) = \frac{1}{6} + c_1 + c_2 = 0$, and $y'(0) = \frac{1}{6} - c_1 - 2c_2 = 3$. Then $c_1 = -\frac{17}{6} - 2c_2$, so $-\frac{16}{6} - c_2 = 0$, and $c_2 = -\frac{8}{3}$. Backsolving, $c_1 = -\frac{17}{6} + \frac{16}{3} = \frac{15}{6} = \frac{5}{2}$, so

$$y = \frac{1}{6}e^x + \frac{5}{2}e^{-x} - \frac{8}{3}e^{-2x}.$$

■

3.5.34. Solve the initial value problem $y'' + y = \cos x$, $y(0) = 1$, $y'(0) = -1$.

Solution: We determine the complementary solution y_c and a particular solution y_p. Since the characteristic equation of the DE is $r^2 + 1 = 0$, the roots are $r = \pm i$, so $y_c = c_1\cos x + c_2\sin x$. Since this overlaps with $f(x) = \cos x$, we take our guess for y_p to be $Ax\cos x + Bx\sin x$. Then

$$y_p'' = A(-2\sin x - x\cos x) + B(2\cos x - x\sin x),$$

so $y_p'' + y_p = -2A\sin x + 2B\cos x = \cos x$. Then $A = 0$ and $B = \frac{1}{2}$, so the general solution is

$$y = \frac{1}{2}x\sin x + c_1\cos x + c_2\sin x.$$

Applying the initial conditions with $y' = \frac{1}{2}(\sin x + x\cos x) - c_1\sin x + c_2\cos x$, $y(0) = c_1 = 1$, and $y'(0) = c_2 = -1$. Thus, the solution to the IVP is

$$y = \frac{1}{2}x\sin x + \cos x - \sin x.$$

■

3.5.54. Use the method of variation of parameters to find a particular solution of the DE $y'' + y = \csc^2 x$.

Solution: From above, we see that $y_1 = \cos x$ and $y_2 = \sin x$ are two linearly independent solutions to the homogeneous DE $y'' + y = 0$. Furthermore, their Wronskian is

$$W(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1.$$

Thus, with $f(x) = \csc^2 x$, the formula for variation of parameters gives a particular solution

$$\begin{aligned} y &= -y_1 \int \frac{y_2(x) f(x)}{W(x)}\, dx + y_2 \int \frac{y_1(x) f(x)}{W(x)} \\ &= -\cos x \int \sin x \csc^2 x\, dx + \sin x \int \cos x \csc^2 x\, dx \\ &= -\cos x \int \csc x\, dx + \sin x \int \cot x \csc x\, dx \\ &= -\cos x \ln|\csc x - \cot x| - \sin x \csc x = -\cos x \ln|\csc x - \cot x| - 1. \end{aligned}$$

■

3.5.58. Use the method of variation of parameters to find a particular solution to the DE $x^2 y'' - 4xy' + 6y = x^3$, using that the complementary solution is $y_c = c_1 x^2 + c_2 x^3$.

Solution: We first normalize the DE to $y'' - \frac{4}{x} y' + \frac{6}{x^2} y = x$. Taking $y_1 = x^2$ and $y_2 = x^3$, their Wronskian is

$$W(x) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4 = x^4.$$

Then with $f(x) = x$ from the normalized equation, the formula for variation of parameters gives a particular solution

$$\begin{aligned} y &= -y_1 \int \frac{y_2(x) f(x)}{W(x)}\, dx + y_2 \int \frac{y_1(x) f(x)}{W(x)} \\ &= -x^2 \int \frac{x^3 x}{x^4}\, dx + x^3 \int \frac{x^2 x}{x^4} \\ &= -x^2 \int dx + x^3 \int \frac{1}{x}\, dx \\ &= -x^3 + x^3 \ln x. \end{aligned}$$

Since the $-x^3$ is already a part of the complementary solution, we may eliminate it and instead take only $y = x^3 \ln x$ as a particular solution to the DE. ■