Homework #4 Solutions

Problems

- Section 2.1: 10, 12, 24
- Section 2.2: 2, 6, 12, 22
- Section 2.3: 2, 12, 20, 24, 30

2.1.10. Suppose that the fish population $P(t)$ in a lake is attacked by a disease at time $t = 0$, with the result that the fish cease to reproduce (so that the birth rate is $\beta = 0$) and the death rate δ (deaths per week per fish) is thereafter proportional to $1/\sqrt{P}$. If there were initially 900 fish in the lake and 441 were left after 6 weeks, how long did it take all the fish in the lake to die?

Solution: We first determine the DE describing the growth rate of the fish. A subtle point is that the death rate factor δ is to be multiplied by P to get the death rate (in terms of fish per week), so this term is $-\delta P = -k\frac{1}{\sqrt{P}}P = -k\sqrt{P}$. Since the birth rate is 0, the DE is $P' = -k\sqrt{P}$. Fortunately, this DE is separable (and indeed autonomous), so we separate and integrate:

$$\frac{1}{\sqrt{P}}P' = -k \qquad \int \frac{1}{\sqrt{P}}\,dP = \int -k\,dt \qquad 2\sqrt{P} = -kt + C.$$

Applying the initial condition $P(0) = 900$, $2(30) = 0 + C$, so $C = 60$. At $t = 6$, $P = 441 = 21^2$, so $2(21) = 60 - 6k$. Then $6k = 18$, so $k = 3$. Finally, we set $P = 0$ and solve for t: $2(0) = 60 - 3t$, so $t = 60/3 = 20$. Hence, the fish all die after 20 weeks. ∎

2.1.12. The time rate of change of an alligator population P in a swamp is proportional to the square of P. The swamp contained a dozen alligators in 1988, and two dozen in 1998. When will there be four dozen alligators in the swamp? What happens thereafter?

Solution: The DE governing the alligator population is $P' = kP^2$, and we have the population data $P(0) = 12$ and $P(10) = 24$, taking 1988 as $t = 0$. Separating this DE and integrating,

$$\frac{1}{P^2}P' = k, \qquad \int \frac{1}{P^2}\,dP = \int k\,dt \qquad -\frac{1}{P} = kt + C.$$

At $t = 0$, $P = 12$, so $-\frac{1}{12} = C$, and $C = -\frac{1}{12}$. Then at $t = 10$, $P = 24$, so $-\frac{1}{24} = 10k - \frac{1}{12}$. Then $10k = \frac{1}{24}$, so $k = \frac{1}{120}$. We therefore have that $\frac{1}{P} = \frac{1}{12} - \frac{1}{240}t = \frac{20-t}{240}$, so

$$P(t) = \frac{240}{20 - t}.$$

We compute when $P(t) = 48$: $48 = \frac{240}{20-t}$, so $20 - t = \frac{240}{48} = 5$, and $t = 15$. Thus, the population will double again by 2003.

Past that, we observe that $P(t) \to +\infty$ as $t \to 20^-$, so as we approach the year 2008, we expect an unlimited number of alligators in the population. ∎

2.1.24. Suppose that a community contains 15,000 people who are susceptible to Michaud's syndrome, a contagious disease. At time $t = 0$, the number $N(t)$ of people who have developed Michaud's syndrome is 5000 and is increasing at the rate of 500 per day. Assume that $N'(t)$ is proportional to the product of the number of those who have caught the disease and those who have not. How long will it take for another 5000 people to develop Michaud's syndrome?

Solution: Given the modeling assumptions, $N(t)$ is goverened by the logistic DE $N' = kN(M - N)$, where $M = 15{,}000$ is the total population and k is a parameter to be determined from the initial data. At $t = 0$, $N_0 = N(0) = 5000$. Since $N'(0) = 500$ infections per day,

$$500 = k(5000)(15{,}000 - 5000) = (5000)(10{,}000)k \quad \Rightarrow \quad k = \frac{1}{100{,}000}.$$

Fortunately, we already have an explicit solution to this DE,

$$N(t) = \frac{MN_0}{N_0 + (M - N_0)e^{-kMt}},$$

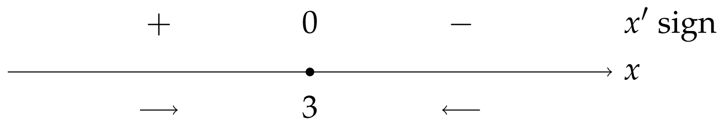
so we set $N(t) = 10{,}000$ and solve for t. Then

$$10{,}000 = \frac{(15{,}000)(5000)}{5000 + 10{,}000e^{-kMt}} = \frac{15{,}000}{1 + 2e^{-kMt}},$$

so $1 + 2e^{-kMt} = \frac{3}{2}$, $-kMt = \ln\frac{1}{4} = -\ln 4$, and, since $kM = \frac{15{,}000}{100{,}000} = \frac{3}{20}$, $t = \frac{20}{3}\ln 4 \approx 9.24$. Therefore, it should take slightly more than 9 days for the next 5000 people to become infected. ∎

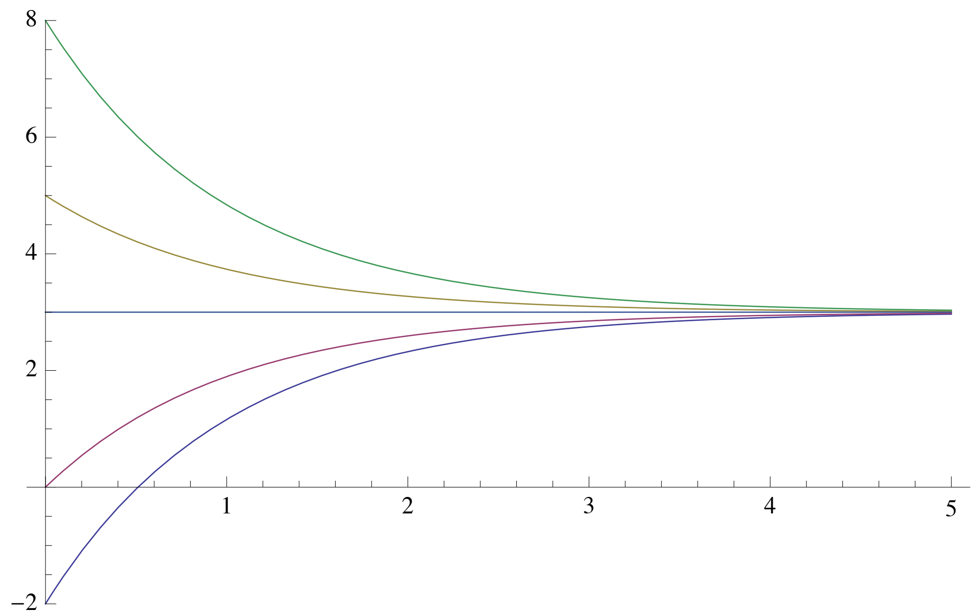
2.2.2. Consider the DE $\frac{dx}{dt} = 3 - x$, letting $f(x) = 3 - x$. Solve the equation $f(x) = 0$ to find the critical points of the autonomous DE. Analyze the sign of $f(x)$ to determine whether each critical point is stable or unstable, and construct the corresponding phase diagram for the DE. Next, solve the DE explicitly for $x(t)$ in terms of t. Finally, use either the exact solution or a computer-generated slope field to sketch solution curves for the given DE, and verify visually the stability of each critical point.

Solution: We first compute the critical points of f: $3 - x = 0$, so $x = 3$ is the only one. Since $f(x)$ is positive for $x < 3$ and negative for $x > 3$, this critical point has the following phase diagram,



and is therefore a stable equilibrium.

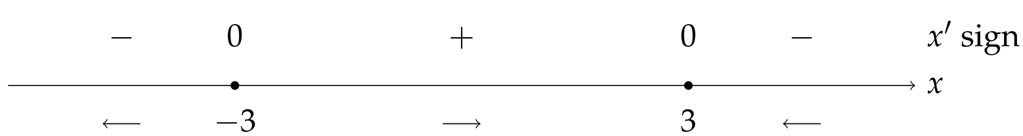
We find its exact solution. This DE is linear, which we rewrite as $x' + x = 3$. Then the integrating factor is $\mu(t) = e^t$, so $(e^t x)' = 3e^t$. Integrating and dividing by $\mu(t)$, $x(t) = 3 - Ce^{-t}$. We plot several of these solutions below:



As expected, these solutions all converge towards the equilibrium $x = 3$. ■

2.2.6. Consider the DE $\frac{dx}{dt} = 9 - x^2$, letting $f(x) = 9 - x^2$. Solve the equation $f(x) = 0$ to find the critical points of the autonomous DE. Analyze the sign of $f(x)$ to determine whether each critical point is stable or unstable, and construct the corresponding phase diagram for the DE. Next, solve the DE explicitly for $x(t)$ in terms of t. Finally, use either the exact solution or a computer-generated slope field to sketch solution curves for the given DE, and verify visually the stability of each critical point.

Solution: We first compute the critical points of f: $9 - x^2 = 0$, so $x^2 = 9$. Then $x = 3$ and $x = -3$ are the equilibria. We note that $f(x)$ is positive between -3 and 3 and negative otherwise, so we obtain the following phase diagram:

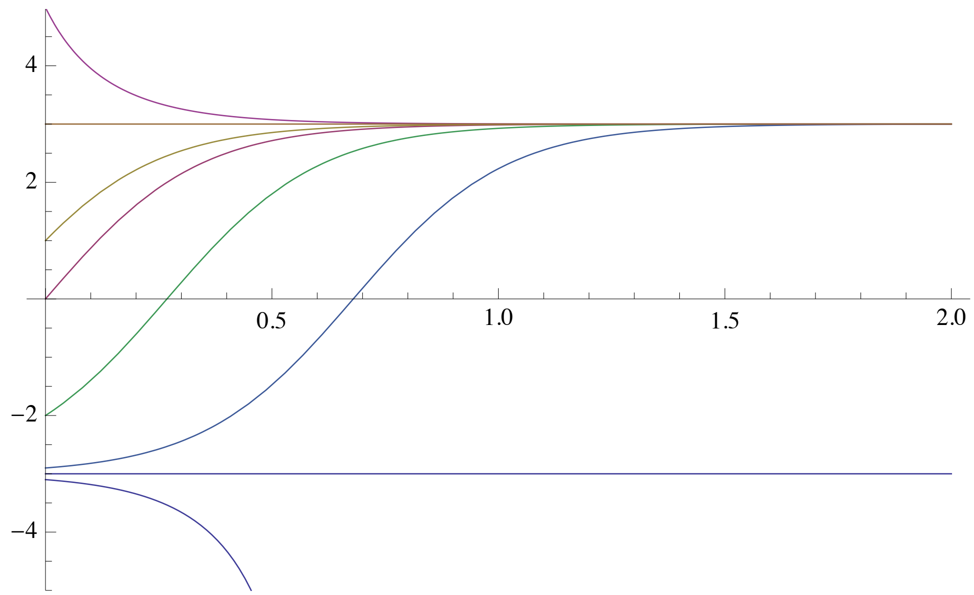


Thus, $x = -3$ is an unstable equilibrium, and $x = 3$ is a stable one.

We find the exact solution to the DE, which separates into $\frac{1}{9-x^2}x' = 1$. We decompose the left-hand side into partial fractions as $\frac{1}{6}(\frac{1}{3+x} + \frac{1}{3-x})$, so, multiplying by 6,

$$\int \frac{1}{3+x} + \frac{1}{3-x}\,dx = \int 6\,dt \quad \Rightarrow \quad \ln|3+x| - \ln|3-x| = 6t + C$$

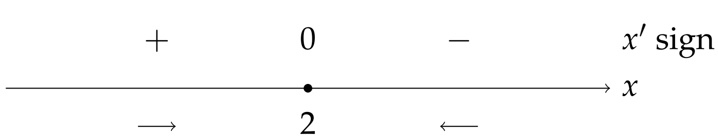
Then $Ce^{-6t} = \frac{3-x}{3+x} = \frac{6}{3+x} - 1$, so $x(t) = \frac{6}{1+Ce^{-6t}} - 3$. Setting $C = 0$ yield the stable equilibrium $x = 3$, but this general solution omits the singular solution $x = -3$ (which in some sense corresponds to $C = +\infty$). We plot several of these solutions below:



As expected, the solutions starting above $x = -3$ all converge towards the equilibrium $x = 3$, and the solutions starting below that diverge to $-\infty$ (in finite time, actually). ■

2.2.12. Consider the DE $\frac{dx}{dt} = (2-x)^3$, letting $f(x) = (2-x)^3$. Solve the equation $f(x) = 0$ to find the critical points of the autonomous DE. Analyze the sign of $f(x)$ to determine whether each critical point is stable or unstable, and construct the corresponding phase diagram for the DE. Next, solve the DE explicitly for $x(t)$ in terms of t. Finally, use either the exact solution or a computer-generated slope field to sketch solution curves for the given DE, and verify visually the stability of each critical point.

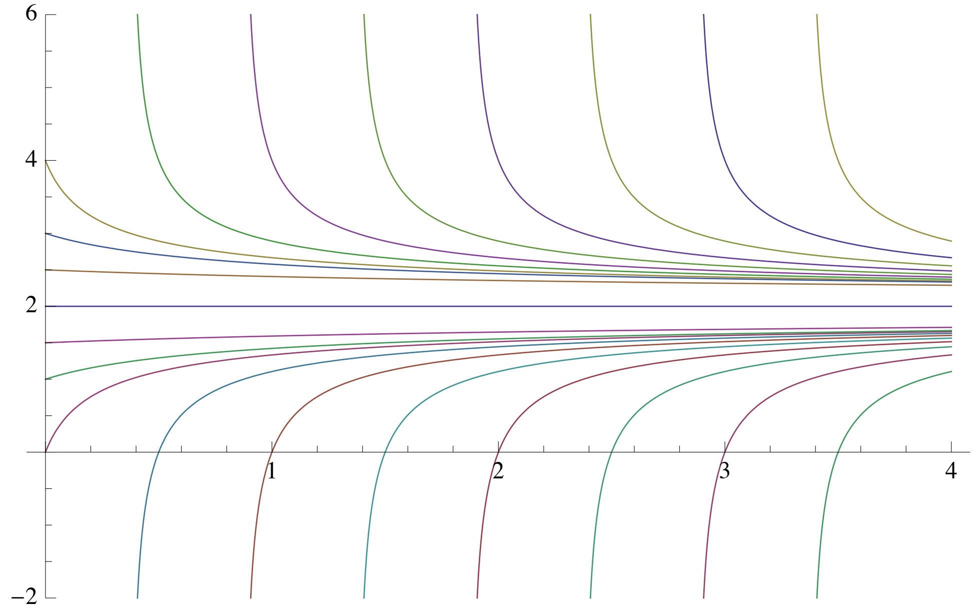
Solution: Solving $(2-x)^3$, we see that $x = 2$ is a triple root, and the only solution. Since $f(x)$ is positive for $x < 2$ and negative for $x > 2$, the critical point $x = 2$ is a stable equilibrium, as we see on the phase diagram:



We solve this separable DE exactly: separating, $(2-x)^{-3}x' = 1$, so integrating yields $\frac{1}{2(2-x)^2} = t + C$. Then $2 - x = \pm\frac{1}{\sqrt{2(t+C)}}$, so $x = 2 \pm \frac{1}{\sqrt{2(t+C)}}$. We note this is defined for $t > -C$, so we let $t_0 = -C$ denote this "blow-up" time; then the solutions are

$$x(t) = 2 \pm \frac{1}{\sqrt{2(t-t_0)}},\ t > t_0$$

and the singular solution $x = 2$. We plot several of these solutions below:



As expected, these solutions all converge towards the stable equilibrium $x = 2$. ■

2.2.22. Consider the DE $dx/dt = x + kx^3$ containing the parameter k. Analyze (as in problem 21) the dependence of the number and nature of the critical points on the value k, and construct the corresponding bifurcation diagram.

Solution: We solve for the critical points of $f(x)$ with $x + kx^3 = 0$. Then $x = 0$ or $1 + kx^2 = 0$, which yields $x = \pm\frac{1}{\sqrt{-k}}$ for $k < 0$. Hence, we see two qualitatively different cases for equilibria: for $k \geq 0$, there is a single equilibrium, $x = 0$, and for $k < 0$ there are two additional equilibria, $x = 1/\sqrt{-k}$ and $x = -1/\sqrt{-k}$.

We also determine the stability of these equilibria. For $k \geq 0$, $f(x) = x + kx^3$ is monotonic, with positive values for $x > 0$ and negative ones for $x < 0$. Hence, the sole equilibrium $x = 0$ is unstable.

For the case when $k < 0$, we make things easier by developing a "first derivative test" for stability. Consider $f'(x) = \frac{df}{dx}$ at a point c where $f(c) = 0$.

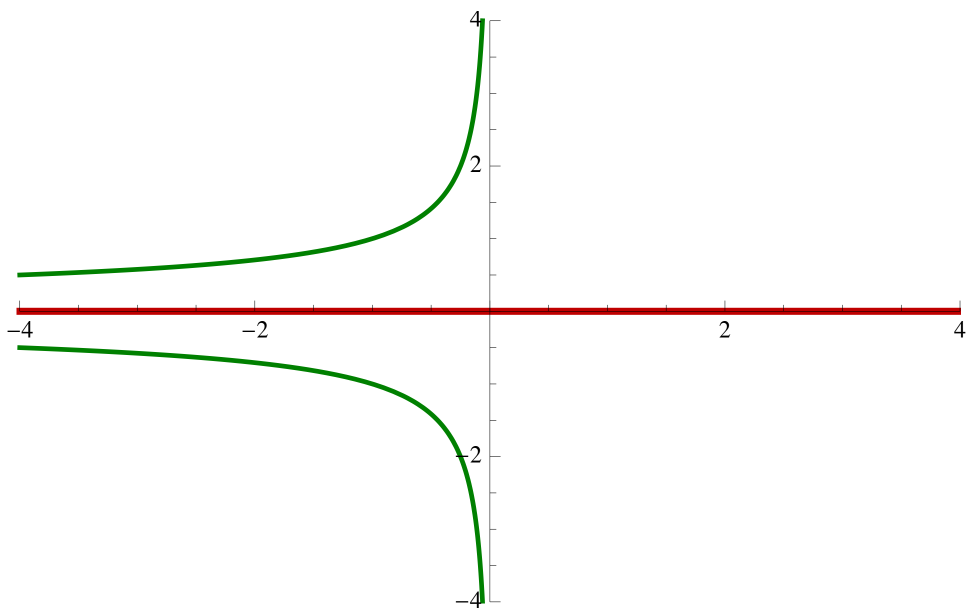
- If $f'(c)$ is positive, then $f(x)$ is positive for x slightly greater than c, and $f(x)$ is negative for x slightly less than c, so $x = c$ must be an unstable equilibrium.

- Conversely, if $f'(c)$ is negative, $f(x)$ changes from positive to negative across $x = c$, so $x = c$ is a stable equilibrium.

Hence, we check the sign of $f'(x) = 1 + 3kx^2$ at the three equilibria:

- At $x = 0$, $f'(0) = 1 + 0 = 1$, which is positive, so $x = 0$ is (still) unstable.
- At $x = \pm 1/\sqrt{-k}$, $f'(\pm 1/\sqrt{-k}) = 1 + \frac{3k}{-k} = 1 - 3 = -2$, which is negative, so these are both stable equilibria.

The bifurcation diagram is plotted below, with stable branches in green and unstable ones in red:



■

2.3.2. Suppose that a body moves through a resisting medium with resistance proportional to its velocity v, so that $dv/dt = -kv$.

(a) Show that its velocity and position at time t are given by $v(t) = v_0 e^{-kt}$ and $x(t) = x_0 + \frac{v_0}{k}(1 - e^{-kt})$.

(b) Conclude that the body travels only a finite distance, and find that distance.

Solution (a): This equation is linear, with integrating factor $\mu(t) = e^{kt}$. Then $(e^{kt}v)' = 0$, so $e^{kt}v = C$, and $v = Ce^{-kt}$. Applying the IC $v(0) = v_0$ gives $C = v_0$, so $v(t) = v_0 e^{-kt}$.

Integrating $v(t)$ from $t = 0$ to t gives $x(t) - x_0$:

$$x(t) - x_0 = \int_0^t v_0 e^{-kt} = -\frac{v_0}{k}(e^{-kt} - 1) = \frac{v_0}{k}(1 - e^{-kt}),$$

so $x(t) = x_0 + \frac{v_0}{k}(1 - e^{-kt})$. ■

Solution (b): Taking $\lim_{t\to\infty} x(t)$, we see that the e^{-kt} term goes to 0, so the limiting x-value is $x_0 + \frac{v_0}{k}$. This is the finite position to which the body travels; the actual distance covered is only $\frac{v_0}{k}$. ■

2.3.12. It is proposed to dispose of nuclear wastes—in drums with weight 640 lb and volume 8 ft^3—by dropping them into the ocean ($v_0 = 0$). The force equation for a drum falling through water is

$$m\frac{dv}{dt} = -W + B + F_R,$$

where the buoyant force B is equal to the weight (at 62.5 lb/ft^3) of the volume of water displaced by the drum (Archimedes's principle) and F_R is the force of water resistance, found empirically to be 1 lb for each foot per second of the velocity of a drum. If the drums are likely to burst upon an impact of more than 75 ft/s, what is the maximum depth to which they can be dropped in the ocean without likelihood of bursting?

Solution: We compute some of the quantities in this model, working in foot-pound-second (fps) units. The buoyancy force is $62.5(8) = 500$ lb, and the weight is $W = 640$ lb. The mass of the barrel is $m = W/g = 640/32 = 20$ slug, where the slug (or lb-s^2/ft) is the fps unit of mass. Finally, $F_R = -(1)v = -v$, because of the empirically determined water resistance constant, so the DE is

$$20v' = -140 - v.$$

This DE is linear, normalizing to $v' + \frac{1}{20}v = 7$, so with integrating factor $\mu(t) = e^{t/20}$, $(e^{t/20}v)' = 7e^{t/20}$, $e^{t/20}v = -140e^{t/20} + C$, and $v = -140 + Ce^{-t/20}$. Since $v(0) = 0$, $C = 140$, and $v(t) = 140(e^{-t/20} - 1)$.

We determine the time and distance where v reaches -75 ft/s:

$$-75 = 140(e^{-t/20} - 1) \quad \Rightarrow \quad e^{-t/20} = \frac{65}{140} = \frac{13}{28} \quad \Rightarrow \quad t = 20\ln\frac{28}{13} \approx 15.345 \text{ s}$$

Integrating $v(t)$ and applying $y_0 = 0$,

$$y(t) = \int_0^t 140(e^{-t/20} - 1)\,dt = 140(20(1 - e^{-t/20}) - t).$$

At $t = 20\ln\frac{28}{13}$, $y = 140(\frac{75}{7} - 20\ln\frac{28}{13}) \approx -648$ ft. So 648 ft is the maximum depth to which the barrels can be dropped without bursting from the impact. ∎

2.3.20. An arrow is shot straight upwards from the ground with an initial velocity of 160 ft/s. It experiences both the deceleration of gravity and deceleration $v^2/800$ due to air resistance. How high in the air does it go?

Solution: We use the solution to the upwards-moving v^2-resistance model given on p. 103 of the text:

$$v(t) = \sqrt{\frac{g}{\rho}}\tan(C_1 - t\sqrt{\rho g}), \quad C_1 = \tan^{-1}(v_0\sqrt{\frac{\rho}{g}}).$$

Since we are given accelerations directly, the DE in the model is $v' = -32 - \frac{1}{800}v^2$, in fps units, with $v_0 = 160$. Then $\rho = 1/800$, and $g = 32$, so

$$v_0\sqrt{\frac{\rho}{g}} = 160\sqrt{\frac{1}{800(32)}} = \frac{160}{160} = 1,$$

a lucky break. Then $C_1 = \tan^{-1}(1) = \pi/4$. We can then solve for the t-value T at which $v = 0$: this occurs when $C_1 = T\sqrt{\rho g} = 0$, or $T = \frac{1}{\sqrt{\rho g}}C_1$. Since $\rho g = \frac{32}{800} = \frac{1}{25}$, $T = 5\pi/4$.

Finally, we use this T value to calculate the arrow's height y. Since $y_0 = 0$,

$$y(T) = \frac{1}{\rho}\ln\left|\frac{\cos(C_1 - T\sqrt{\rho g})}{\cos C_1}\right|.$$

By construction, $C_1 - T\sqrt{\rho g} = 0$, so its cosine is 1, and $\cos C_1 = \cos \pi/4 = 1/\sqrt{2}$, so

$$y(T) = \frac{1}{\rho}\ln\sqrt{2} = \frac{800}{2}\ln 2 = 400\ln 2 \approx 277.26 \text{ ft.}$$

■

2.3.24. The mass of the sun is 329,320 times that of the earth and its radius is 109 times that of the earth.

(a) To what radius (in meters) would the earth have to be compressed in order for it to become a *black hole*, with the escape velocity from its surface equal to the velocity $c = 3 \times 10^8$ m/s of light?

(b) Repeat part (a) with the sun in place of the earth.

Solution (a): We compute the radius R so that the escape velocity $v_e = \sqrt{\frac{2GM}{R}} = c$. Then $c^2 = \frac{2GM}{R}$, so $R = \frac{2GM}{c^2}$. (This distance is known in physics as the *Schwarzchild radius*.) Using $M_e = 5.975 \times 10^{24}$ kg for the mass of the earth and the given values for G and c,

$$R_e = \frac{2(6.67 \times 10^{-11})(5.975 \times 10^{24})}{(3 \times 10^8)^2} \approx 8.86 \text{ mm.}$$

■

Solution (b): The radius R is actually proportional to the mass M, so the Schwarzchild radius of the sun is 329,320 that of Earth's:

$$R_s = 329{,}320(8.86 \text{ mm}) \approx 2.92 \text{ km.}$$

■

2.3.30. In Jules Verne's original problem, the projectile launched from the surface of the earth is attracted by both the earth and the moon, so its distance $r(t)$ from the center of the earth satisfies the IVP

$$\frac{d^2r}{dt^2} = -\frac{GM_e}{r^2} + \frac{GM_m}{(S-r)^2}, \quad r(0) = R, \quad r'(0) = v_0,$$

where M_e and M_m are the masses of the earth and the moon, R is the radius of the earth, and $S = 384{,}400$ km is the distance between the centers of the earth and the moon. To reach the moon, the projectile must only just pass the point between the moon and the earth where its net acceleration vanishes. Thereafter, it is "under the control" of the moon, and falls from there to the lunar surface. Find the *minimal* launch velocity v_0 that suffices for the projectile to make it "From the Earth to the Moon".

Solution: We first calculate the radius R_b at which the gravitational forces of the moon and the earth balance:

$$0 = -\frac{GM_e}{R_b^2} + \frac{GM_m}{(S-R_b)^2} \quad \Rightarrow \quad \frac{(S-R_b)^2}{R_b^2} = \frac{M_m}{M_e} \quad \Rightarrow \quad \frac{S}{R_b} - 1 = \sqrt{\frac{M_m}{M_e}}.$$

Then $R_b = \frac{S\sqrt{M_e}}{\sqrt{M_e}+\sqrt{M_m}}$ (approximately 346,000 km), and $S - R_b = \sqrt{\frac{M_m}{M_e}}R_b = \frac{S\sqrt{M_m}}{\sqrt{M_e}+\sqrt{M_m}}$. We now calculate the initial velocity required to reach this balance point from the surface of the earth. The second-order DE govering the radius is

$$r'' = -\frac{GM_e}{r^2} + \frac{GM_m}{(S-r)^2},$$

so we convert it to a first-order DE in $v(r) = \frac{dr}{dt}$, with $r'' = v\frac{dv}{dr} = vv'$. Integrating from $r = R$ to $r = R_b$, at which point we wish $v = 0$, we have

$$-\frac{1}{2}v_0^2 = \left[\frac{GM_e}{r} + \frac{GM_m}{S-r}\right]_{r=R}^{r=R_b} = \frac{GM_e}{R_b} + \frac{GM_m}{S-R_b} - \frac{GM_e}{R} - \frac{GM_m}{S-R}.$$

Multiplying by -2 and substituting in our expressions for R_b and $S - R_b$,

$$\begin{aligned} v_0^2 &= 2G\left(\frac{M_e}{R} + \frac{M_m}{S-R} - \frac{M_e}{R_b} - \frac{M_m}{S-R_b}\right) \\ &= 2G\left(\frac{M_e}{R} + \frac{M_m}{S-R} - \frac{\sqrt{M_e}(\sqrt{M_e}+\sqrt{M_m})}{S} - \frac{\sqrt{M_m}(\sqrt{M_e}+\sqrt{M_m})}{S}\right) \\ &= 2G\left(\frac{M_e}{R} + \frac{M_m}{S-R} - \frac{(\sqrt{M_e}+\sqrt{M_m})^2}{S}\right) \end{aligned}$$

Plugging in values for M_e, M_m, R, S, and G from the text, we determine that the required initial velocity is

$$v_0 = \sqrt{2G\left(\frac{M_e}{R} + \frac{M_m}{S-R} - \frac{(\sqrt{M_e}+\sqrt{M_m})^2}{S}\right)} \approx 11{,}067\ \frac{\text{m}}{\text{s}}.$$

Rounding to significant figures, this is 11.1 km/s. ■