Homework #3 Solutions

Problems

- Section 1.6: 8, 22, 28, 34, 36, 44, 46, 60.
- Section 2.1: 16.
- Extra Problem #1:

1.6.8. Find the general solution of the DE $x^2y' = xy + x^2e^{y/x}$.

Solution: Dividing by x^2, we see that the DE becomes $y' = \frac{y}{x} + e^{y/x}$, a function of y/x alone. Let $v = y/x$, so that $y = xv$, and $y' = xv' + v$. Then the DE becomes

$$xv' + v = v + e^v,$$

which simplifies and separates to $e^{-v}v' = \frac{1}{x}$. Integrating, $-e^{-v} = \ln|x| + C$, so $v = -\ln(C - \ln|x|)$. Backsolving for y, $y = xv = -x\ln(C - \ln|x|)$ is the general solution to the DE. ∎

1.6.22. Find the general solution of the DE $x^2y' + 2xy = 5y^4$.

Solution: We observe that this is a Bernoulli equation with $n = 4$, so we make the substitution $v = y^{-3}$. Then $y = v^{-1/3}$, so $y' = -\frac{1}{3}v^{-4/3}v'$. Thus the DE becomes

$$-\frac{1}{3}x^2v^{-4/3}v' + 2xv^{-1/3} = 5v^{-4/3}.$$

Dividing by the coefficient $-\frac{1}{3}x^2v^{-4/3}$ on v', this DE becomes $v' - \frac{6}{x}v = -\frac{15}{x^2}$, which is linear. The coefficient function on v is $P(x) = -\frac{6}{x}$, so the integrating factor is then $\mu(x) = e^{-6\ln|x|} = x^{-6}$. Multiplying by $\mu(x)$, we have

$$(x^{-6}v)' = x^{-6}v' - 6x^{-7}v = -15x^{-8},$$

so upon integrating, $x^{-6}v = \frac{15}{7}x^{-7} + C$. Then $v = \frac{15}{7x} + Cx^6 = \frac{15+Cx^7}{7x}$, rescaling C. Finally, $v = y^{-3}$, so the general solution is

$$y = \sqrt[3]{\frac{7x}{15 + Cx^7}}.$$

∎

1.6.28. Find the general solution of the DE $xe^y y' = 2(e^y + x^3 e^{2x})$.

Solution: We observe that y occurs only in e^y, so we let $v = e^y$. Then $y = \ln v$, so $y' = \frac{1}{v}v'$. Hence, the DE becomes

$$xv\frac{1}{v}v' = 2(v + x^3 e^{2x}),$$

so $v' - \frac{2}{x}v = 2x^2 e^{2x}$, which is linear in v. The integrating factor is $\mu(x) = e^{-2\ln|x|} = x^{-2}$, so then

$$(x^{-2}v)' = 2e^{2x}.$$

Integrating, $x^{-2}v = e^{2x} + C$, so $v = x^2 e^{2x} + Cx^2$. Finally, the general solution is $y = \ln v = \ln(x^2 e^{2x} + Cx^2)$. ∎

1.6.34. Verify that the DE $(2xy^2 + 3x^2)\,dx + (2x^2 y + 4y^3)\,dy = 0$ is exact, and find its general solution.

Solution: Letting $M(x, y) = 2xy^2 + 3x^2$ and $N(x, y) = 2x^2 y + 4y^3$, we check the exactness condition $M_y = N_x$. Since $M_y = 4xy + 0 = 4xy$ and $N_x = 4xy + 0 = 4xy$, the condition is satisfied, and the DE is exact.

We find F first by integrating M with respect to x:

$$F(x, y) = \int 2xy^2 + 3x^2\,dx = x^2 y^2 + x^3 + g(y),$$

where g is some unknown function of y. We compute F_y and compare it to N: $F_y = 2x^2 y + g'(y) = N = 2x^2 y + 4y^3$, so $g'(y) = 4y^3$. Integrating with respect to y, $g(y) = y^4 + C$. Thus, the general solution is given implicitly by the equation $x^2 y^2 + x^3 + y^4 = C$. ∎

1.6.36. Verify that the DE $(1 + ye^{xy})\,dx + (2y + xe^{xy})\,dy = 0$ is exact, and find its general solution.

Solution: Letting $M(x, y) = 1 + ye^{xy}$ and $N(x, y) = 2y + xe^{xy}$, we check the exactness condition $M_y = N_x$. Since $M_y = 0 + y(xe^{xy}) = xye^{xy}$ and $N_x = 0 + x(ye^{xy}) = xye^{xy}$, the condition is satisfied, and the DE is exact.

We find F first by integrating M with respect to x:

$$F(x, y) = \int 1 + ye^{xy}\,dx = x + e^{xy} + g(y),$$

where g is some unknown function of y. We compute F_y and compare it to N: $F_y = ye^{xy} + g'(y) = N = 2y + xe^{xy}$, so $g'(y) = 2y$. Then $g(y) = y^2 + C$, so the general solution to F is $F(x, y) = x + e^{xy} + y^2 + C$. Hence, the general solution to the DE is given implicitly by the equation

$$x + e^{xy} + y^2 = C$$

for different values of C. ∎

1.6.44. Find a general solution of the reducible second-order equation $yy'' + (y')^2 = 0$. Assume x, y, and/or y' to be positive if helpful, but state your assumptions.

Solution: Since this second-order DE does not have any explicit dependence on x, it is indeed reducible. Let $p(y) = y'$; then $y'' = pp'$, and the DE becomes $ypp' + p^2 = 0$. Normalizing, $p' = -p/y$, or $p' + \frac{1}{y}p = 0$, which is linear. Using the integrating factor $\mu(y) = e^{\ln y} = y$, $(yp)' = 0$, so $yp = C$, and $p(y) = C/y$.

Since $y' = p$, this gives the separable equation $y' = C/y$, so $yy' = C$. Integrating with respect to x, $\frac{1}{2}y^2 = Cx + D$, so $y = \sqrt{Cx + D}$ (rescaling C and D) is the general solution.■

1.6.46. Find a general solution of the reducible second-order equation $xy'' + y' = 4x$. Assume x, y, and/or y' to be positive if helpful, but state your assumptions.

Solution: Since this second-order DE does not have any explicit dependence on x, it is indeed reducible. Let $p(x) = y'$, so that $y'' = p'$. Then the DE is $xp' + p = 4x$, which is linear; in fact, the left-hand side is already $(xp)'$. Integrating, $xp = 2x^2 + C$, so $p = 2x + \frac{C}{x}$.

We therefore have that $y' = 2x + \frac{C}{x}$, so we integrate to obtain y:

$$y = \int 2x + \frac{C}{x}\,dx = x^2 + C\ln|x| + D.$$

■

1.6.60. Solve the DE

$$\frac{dy}{dx} = \frac{2y - x + 7}{4x - 3y - 18}$$

first by finding h and k so that the substitutions $x = u + h$, $y = v + k$ transform it into a homogeneous DE.

Solution: We first convert this DE into a homogeneous DE in new variables u and v, with $x = u + h$ and $y = v + k$ for some constants h, k. Making these substitutions, $\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{du}\frac{du}{dx} = \frac{dv}{du}$, since $\frac{dy}{dv} = 1$ and $\frac{du}{dx} = 1$, and

$$\frac{2y - x + 7}{4x - 3y - 18} = \frac{2(v + k) - (u + h) + 7}{4(u + h) - 3(v + k) - 18} = \frac{2v - u + 2k - h + 7}{4u - 3v + 4h - 3k - 18}.$$

In order for the DE to be homogeneous, the constant terms $2k - h + 7$ and $4h - 3k - 18$ in this fraction must both be 0. We use this to solve for h and k: $h = 2k + 7$, so $8k + 28 - 3k - 18 = 0$. Then $5k = -10$, so $k = -2$, and $h = 3$. Thus, with the substitutions $x = u + 3$ and $y = v - 2$, the DE becomes

$$\frac{dv}{du} = \frac{2v - u}{4u - 3v} = \frac{2(v/u) - 1}{4 - 3(v/u)}.$$

Since this equation is homogeneous, let $z(u) = v/u$, so that $v = uz$. Then $v' = uz' + z$, so the DE becomes

$$uz' = \frac{2z-1}{4-3z} - z = \frac{2z-1+3z^2-4z}{4-3z} = \frac{3z^2-2z-1}{4-3z},$$

which is separable. Upon separating, we have $\frac{4-3z}{3z^2-2z-1}z' = \frac{1}{u}$; to integrate the left-hand side, we expect to use partial fractions. Fortunately, $3z^2-2z-1$ factors as $(3z+1)(z-1)$, so we need to solve the functional equation $A(3z+1) + B(z-1) = 4-3z$ for A and B. Then $3A + B = -3$ and $A - B = 4$, so $A = \frac{1}{4}$ and $B = -\frac{15}{4}$. Hence, integrating both sides,

$$\int \frac{4-3z}{3z^2-2z-1}\,dz = \frac{1}{4}\int \frac{1}{z-1} - \frac{15}{3z+1}\,dz = \frac{1}{4}\left(\ln|z-1| - 5\ln|3z+1|\right) = \ln|u| + C.$$

Multiplying by 4 and exponentiating, $(z-1)(3z+1)^{-5} = Cu^4$, so

$$(z-1) = Cu^4(3z+1)^5.$$

Backsubstituting $z = v/u$ to eliminate z, we have

$$\frac{1}{u}(v-u) = Cu^4\frac{1}{u^5}(3v+u)^5 = C\frac{1}{u}(3v+u)^5,$$

and muliplying by u gives $v - u = C(3v+u)^5$. Finally, backsubstituting with $u = x-3$ and $v = y+2$, the implicit solution is

$$(y-x+5) = C(3y+x+3)^5,$$

for which there is no hope of solving for y explicitly. ∎

2.1.16. Consider a rabbit population $P(t)$ satisfying the logistic equation $P' = aP - bP^2$. If the initial population is 120 rabbits and there are 8 births per month and 6 deaths per month occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 95% of the limiting population M?

Solution: From Problem 15, we see that the limiting population is $M = B_0P_0/D_0 = 8(120)/6 = 160$ rabbits. At $t = 0$, the net rate of change of P is 2 rabbits/month, so

$$2 = k(120)(160-120) = 4800k.$$

Then $k = 1/2400$, so $kM = 160/2400 = 1/15$. We solve for the T when $P(T) = 0.95M$; then

$$0.95M = \frac{MP_0}{P_0 + (M-P_0)e^{-kMT}},$$

so $(M-P_0)e^{-kMT} = \frac{P_0}{0.95} - P_0$. Substituting in values, $40e^{-T/15} = \frac{120}{0.95} - 120$, so $e^{-T/15} = \frac{3}{0.95} - 3 = \frac{3}{19}$, and $T = 15\ln\frac{19}{3} \approx 27.7$. Hence, it takes the population 27.7 months to reach 95% of its limit. ∎

Extra Problem #1. Consider the DE $(x+y)y' = x - y$.

(a) Solve the DE using the homogeneous substitution $v = y/x$. An implicit solution is acceptable.

(b) We can rearrange the DE into the differential form

$$(y-x)\,dx + (x+y)\,dy = 0.$$

Is this equation exact? If so, find an implicit solution to the equation using our techniques for exact DEs. Show that your solution is equivalent to your answer from part (a). Which method was easier?

Solution (a): Let $v = y/x$, so $y = xv$, and $y' = xv' + v$. Then the DE becomes

$$xv' + v = \frac{1-v}{1+v} \quad \Rightarrow \quad xv' = \frac{1-v}{1+v} - v = \frac{1-v-v^2-v}{1+v} = \frac{1-2v-v^2}{1+v}.$$

Separating variables, $\frac{1+v}{1-2v-v^2}v' = \frac{1}{x}$. Fortunately, the left-hand side is integrable, since $(1-2v-v^2)' = (-2-2v)v' = -2(1+v)v'$, so we have

$$-\frac{1}{2}\ln|1-2v-v^2| = \ln|x| + C \quad \Rightarrow \quad \ln|1-2v-v^2| + \ln(x^2) = C.$$

Combining the logs and exponentiating, $x^2(1-2v-v^2) = C$, or $x^2 - 2xy - y^2 = C$. ■

Solution (b): Since $M(x,y) = y - x$ and $N(x,y) = x + y$, $M_y = 1$ and $N_x = 1$. Therefore, the DE is exact. To find the function $F(x,y)$ defining its implicit solutions, integrate $M = F_x$ with respect to x:

$$F(x,y) = \int y - x\,dx = xy - \frac{x^2}{2} + g(y).$$

Then $F_y = x + g'(y) = N = x + y$, so $g'(y) = y$, and $g(y) = \frac{1}{2}y^2$. Hence,

$$F(x,y) = xy - \frac{x^2}{2} + \frac{1}{2}y^2 = C$$

determines implicit solutions to the DE. Multiplying the equation by -2, we obtain $x^2 - 2xy - y^2 = C$, which is exactly the solution from part (a).

This method seems easier, as it required integration of only polynomial functions and no fraction or log manipulations. ■