Final Practice Problems

0. Review the practice materials and solutions for Midterms #1 and #2, as well as the solutions to the homework assignments.

1. Find the general solution, possibly implicit, to each of the following DEs or systems of DEs. If an initial condition is given, also find the particular solution matching it.

(a) $x' = 3x + 4y$, $y' = 3x + 2y$, $x(0) = 1$, $y(0) = 1$

(b) $\left(x^3 + \frac{y}{x}\right) dx + (y^2 + \ln x)\, dy = 0$

(c) $x_1' = -x_1 - 3x_2 + 2x_3$, $x_2' = x_1 + 2x_2 - x_3$, $x_3' = -x_1 - 2x_2 + 2x_3$

(d) $(1 + x)y\, dx + x\, dy = 0$

(e) $x' = -x + 2y + t$, $y' = -x - 4y + 1 + t$

(f) $y' = (1 - y) \cos x$, $y(\pi) = 2$

2. Brine circulates at a rate of $r = 10$ gallons per minute between 3 tanks, from tank 1 to tank 2 to tank 3 and back to tank 1. The tanks have volumes $V_1 = 20$, $V_2 = 50$, and $V_3 = 20$ gallons, and at time t_0 tank 1 contains $x_0 = 18$ pounds of salt, while the other tanks contain none.

(a) Write a system of differential equations that governs the amounts of salt $x_i(t)$ in each tank.

(b) Find the general solution to this system of equations.

(c) Find the solution matching the initial condition stated above.

3. Let $A = \begin{bmatrix} 7 & 4 \\ -1 & 3 \end{bmatrix}$.

(a) Find 2 linearly independent solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ to the system $\mathbf{x}' = A\mathbf{x}$ using eigenvalue and eigenvector techniques. Show that the solutions you determine are actually linearly independent.

(b) Form a fundamental matrix $\Phi(t)$ from these two solutions and use it to compute the matrix exponential e^{At}.

(c) Use that $A = 5I + B$, where B is a nilpotent matrix, to compute e^{At} directly, without requiring the eigenvalues and eigenvectors computed above. Check that you obtain the same answer as in the previous part.

(d) Use variation of parameters to find the unique solution to the nonhomogeneous initial value problem

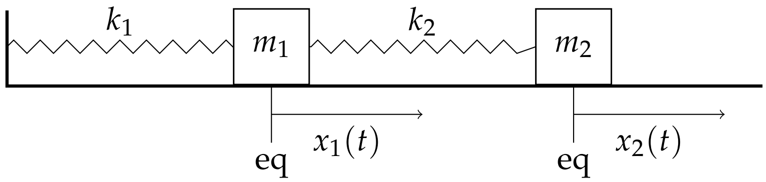
$$\mathbf{x}' = \begin{bmatrix} 7 & 4 \\ -1 & 3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ e^{5t} \end{bmatrix}, \qquad \mathbf{x}(2) = \begin{bmatrix} 7e^{10} \\ -2e^{10} \end{bmatrix}$$

4. Consider the higher-order linear system $x'' - 5x' + 9x - 3y = 0$, $y' + 2y - 5x = 0$.

(a) Introduce the variable $z = x'$ and rewrite this system as a first-order system.

(b) Find the general solution $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$ of the system.

5. Consider the coupled mass-spring system pictured below:



Assume that $m_1 = 2$ kg, $m_2 = 1$ kg, $k_1 = 16$ N/m, $k_2 = 8$ N/m.

(a) Use force diagrams and Newton's second law ($F = ma$) on each mass to write a set of coupled second-order linear DEs describing the displacements $x_1(t)$ and $x_2(t)$.

(b) Using new variables $y_1 = x_1'$ and $y_2 = x_2'$, rewrite the system as a homogeneous first-order system in 4 variables. Find the eigenvalues and eigenvectors of the resulting system and use them to write a general solution to the system.

(c) Find frequencies ω_1, ω_2 and constant vectors $\mathbf{v}_1, \mathbf{v}_2$ so that the general solution to $x_1(t)$ and $x_2(t)$ is in the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \cos(\omega_1 t - \alpha_1)\mathbf{v}_1 + c_2 \cos(\omega_2 t - \alpha_2)\mathbf{v}_2,$$

where the amplitudes c_1, c_2 and the phases α_1, α_2 are parameters depending on the initial values of x_1, x_2, and their derivatives.

6. Each of the following linear or affine-linear systems has a single critical point. Find this critical point and characterize its type and stability.

(a) $x' = 2x - y$, $y' = 3x - 2y$

(b) $x' = 2x - 5y - 1$, $y' = x - 2y - 1$

(c) $x' = -x + 4y$, $y' = -2x + 3y$

(d) $x' = x - 2y$, $y' = 3x - 4y - 2$

7. Find all of the critical points of the autonomous system $x' = (y - x)(1 - x - y)$, $y' = x(2 + y)$. Linearize the system at each critical point to characterize its type and stability, if possible.

8. The linear system $\mathbf{x}' = A\mathbf{x}$ has the general solution

$$\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} 4 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Use this information to recover A. (Hint: $\Phi'(t) = A\Phi(t)$.)