Solutions to Midterm #2 Practice Problems

1. Compute the derivative of each function below. Simplify your answers where possible.

(a) $f(x) = x^3 + \frac{1}{x^3} + \sqrt[3]{x}$

Solution: Since $f(x) = x^3 + x^{-3} + x^{1/3}$, the derivative is $f'(x) = 3x^2 - 3x^{-4} + \frac{1}{3}x^{-2/3}$, which we can rewrite as

$$f'(x) = 3x^2 - \frac{3}{x^4} + \frac{1}{3x^{2/3}}.$$

(b) $h(t) = (4t^2 - t^3)e^t$

Solution:

Using the product rule,

$$h(t) = (4t^2 - t^3)'e^t + (4t^2 - t^3)(e^t)' = (8t - 3t^2)e^t + (4t^2 - t^3)e^t = (8t + t^2 - t^3)e^t.$$

(c) $L(u) = \ln(u)\ln(\ln u)$

Solution: We first observe that $L(u)$ is a product of $f(u) = \ln u$ and $g(u) = \ln(\ln u)$. Then $f'(u) = \frac{1}{u}$, but we must use the chain rule to find $g'(u)$. Let $z = \ln u$; then $g(u) = \ln z$, so

$$g'(u) = \frac{1}{z} \cdot z' = \frac{1}{\ln u} \cdot \frac{1}{u}.$$

Coming back to the derivative of $L(u)$, we compute that

$$L'(u) = f'(u)g(u) + f(u)g'(u) = \frac{1}{u}\ln(\ln u) + \ln u \frac{1}{\ln u} \cdot \frac{1}{u} = \frac{\ln(\ln u) + 1}{u}.$$

(d) $P(z) = \frac{e^{3z}}{z^{3/2}}$

Solution: We use the quotient rule, with $f(z) = e^{3z}$ and $g(z) = z^{3/2}$. Then $f'(z) = 3e^{3z}$ and $g'(z) = \frac{3}{2}z^{1/2}$, so

$$P'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2} = \frac{3e^{3z}z^{3/2} - e^{3z}(\frac{3}{2}z^{1/2})}{(z^{3/2})^2}.$$

We simplify the answer by factoring as much as possible out of the numerator and cancelling with the denominator:

$$P'(z) = \frac{(3z^{3/2} - \frac{3}{2}z^{1/2})e^{3z}}{z^3} = \frac{3(2z-1)(\frac{1}{2}z^{1/2})e^{3z}}{z^3} = \frac{3(2z-1)e^{3z}}{2z^{5/2}}.$$

(e) $Q(w) = e^{w^3 - 2w^2}$

Solution: The function $Q(w)$ is a composite of two functions: letting $z = g(w) = w^3 - 2w^2$ be a new, intermediate variable containing all the ws, we see that $Q(w)$ is the composite of $f(z) = e^z$ and $g(w)$. Hence, we use the chain rule to compute its derivative. First, we compute the derivatives of f and g:

$$f'(z) = e^z \qquad g'(w) = 3w^2 - 2(2w) = 3w^2 - 4w.$$

Then the derivative of $Q(w)$ is

$$Q'(w) = f'(z)g'(w) = e^z(3w^2 - 4w) = (3w^2 - 4w)e^{w^3 - 2w^2}.$$

2. Let $f(x) = 3x^5 - 20x^3$.

(a) Find $f'(x)$ and $f''(x)$.

Solution: Using the power rule, we find that

$$f'(x) = 3(5x^4) - 20(3x^2) = 15x^4 - 60x^2.$$

Taking another derivative, we find that

$$f''(x) = 15(4x^3) - 60(2x) = 60x^3 - 120x.$$

(b) Find the critical points of $f(x)$.

Solution: To find the critical points of $f(x)$, we solve the equation $f'(x) = 0$ for x. (Since $f(x)$ is a polynomial, there are no places where $f'(x)$ is undefined, so we do not get any of that type of critical point.) Hence, we wish to solve

$$15x^4 - 60x^2 = 0.$$

Factoring, we see this is $15x^2(x^2 - 4) = 0$, which factors further as

$$15x^2(x - 2)(x + 2) = 0.$$

Therefore, the roots are $x = 0$, $x = 2$, and $x = -2$, so these are the critical points of $f(x)$.

(c) Characterize each critical point as a local minimum, local maximum, or neither. Justify your answers.

Solution: Since we have computed the second derivative of f to be $f''(x) = 60x^3 - 120x$, we use the second derivative test to get information about these critical points:

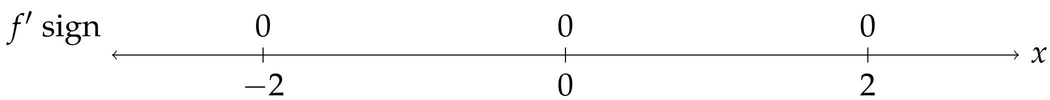
$$\begin{aligned} f''(0) &= 60(0)^3 - 120(0) = 0, \\ f''(2) &= 60(2)^3 - 120(2) = 480 - 240 = 240, \\ f''(-2) &= 60(-2)^3 - 120(-2) = -480 + 240 = -240. \end{aligned}$$

Since $f''(2) > 0$, $f(x)$ has a local minimum at $x = 2$, and since $f''(-2) < 0$, $f(x)$ has a local maximum at $x = -2$. At $x = 0$, however, the second derivative test is inconclusive, and we need to use the first derivative test instead.

For the first derivative test, we check the sign of $f'(x)$ on either side of $x = 0$. The critical points divide up the domain of $f(x)$ so that $f'(x)$ has a single sign on the intervals $(-2, 0)$ and $(0, 2)$. Hence, we pick a point p from each interval and compute $f'(p)$. From $(0, 2)$, we pick $x = 1$, so $f'(1) = 15 - 60 = -45$, and from $(-2, 0)$, we pick $x = -1$, so $f'(-1) = 15(-1)^4 - 60(-1)^2 = 15 - 60 = -45$. Hence, $f'(x)$ is negative on each side of $x = 0$, so we have neither a minimum nor a maximum there.

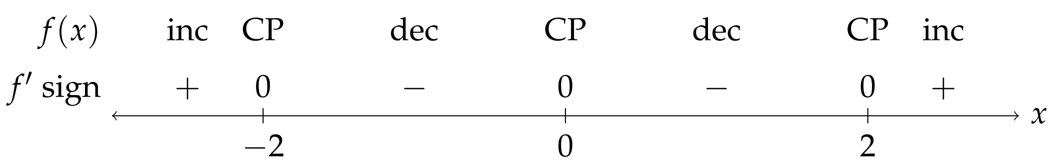
(d) Find the intervals on which $f(x)$ is increasing and on which $f(x)$ is decreasing.

Solution: The critical points divide the real line (the domain of f) into 4 open intervals, $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$, and $(2, \infty)$ as illustrated below:



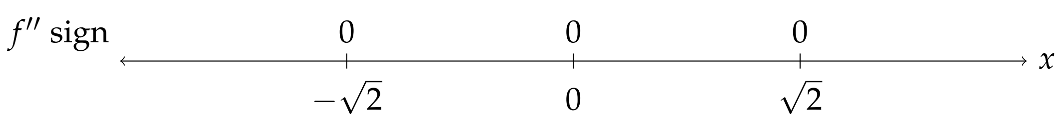
From the first derivative test above, we computed that $f'(x)$ is negative on both $(-2, 0)$ and $(0, 2)$, so $f(x)$ is decreasing on those intervals.

Since $f''(-2) < 0$, $f'(x)$ is decreasing from $+$ to $-$ at $x = -2$, so $f'(x)$ is positive on $(-\infty, -2)$. Likewise, $f''(2) > 0$, so $f'(x)$ increases from $-$ to $+$, and $f'(x)$ is positive on $(2, \infty)$. Therefore, $f(x)$ is increasing on both of these intervals. We illustrate these signs ane behaviors below:



(e) Find the inflection points of $f(x)$. Justify your answers.

Solution: We check where $f''(x) = 0$: $60x^3 - 120x = 0$, so $60x(x^2 - 2) = 0$. Hence, $x = 0$, or $x^2 - 2 = 0$, so $x = \sqrt{2}$ or $x = -\sqrt{2}$. As with the critical points, these three points divide the real line into intervals on which $f''(x)$ is all positive or all negative:



We check the sign of $f''(x)$ on each interval.

- On the interval $(-\infty, -\sqrt{2})$, $f''(-2) < 0$ from the second derivative test computations above, so $f''(x)$ is negative on this interval.
- On $(-\sqrt{2}, 0)$, $f''(-1) = 60$, so $f''(x)$ is positive on this interval.

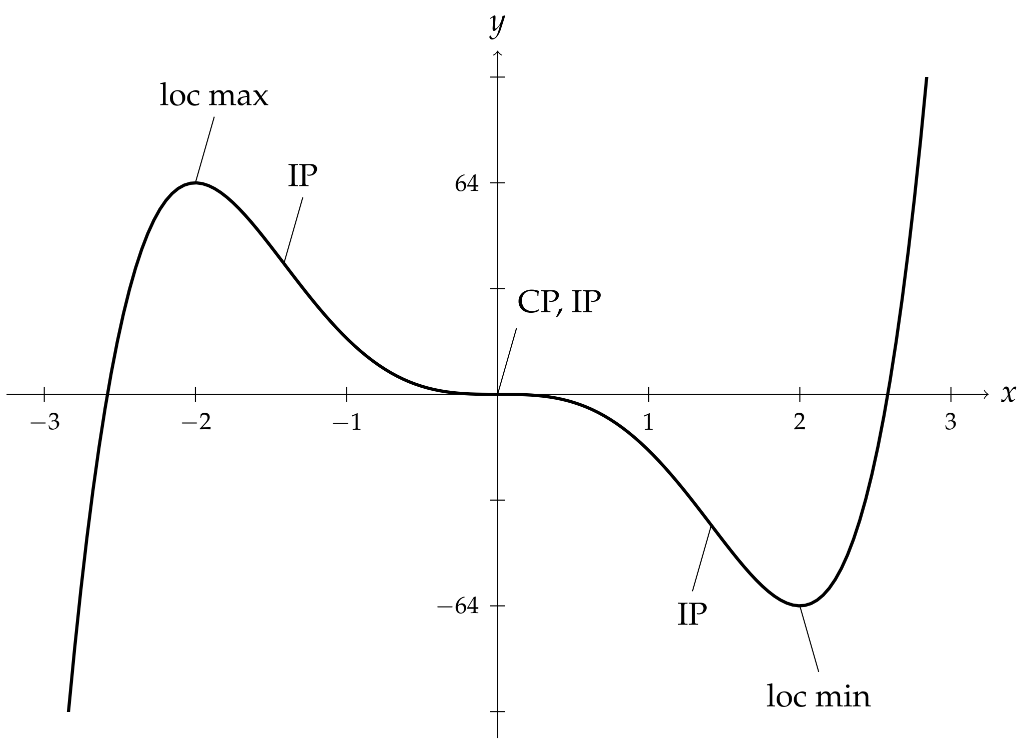
- On $(0, \sqrt{2})$, $f''(1) = -60$, so $f''(x)$ is negative on this interval.
- Finally, on $(\sqrt{2}, \infty)$, $f''(2) > 0$, so $f''(x)$ is positive here.

Since the sign of $f''(x)$ changes at each boundary point, $f(x)$ has an inflection point at all three points.

$f(x)$	cc dn	IP	cc up	IP	cc dn	IP	cc up
f'' sign	$-$	0	$+$	0	$-$	0	$+$
x		$-\sqrt{2}$		0		$\sqrt{2}$	

(f) Use the information in the parts above to make an accurate graph of $f(x)$ on the axes below. Indicate the scale on the x- and y-axes, and label the graph with the local extrema and inflection points.

Solution: We compute the values of $f(x)$ at the local extrema: $f(2) = 3(32) - 20(8) = 96 - 160 = -64$, and $f(-2) = 64$.



3. Let $h(t) = (t^2 - 4)^{2/3}$.

(a) Find $h'(t)$. Simplify your answer.

Solution: Using the chain rule, with outer function $z^{2/3}$ and inner function $t^2 - 4$, we have

$$h'(t) = \frac{2}{3}(t^2 - 4)^{-1/3}(2t) = \frac{4t}{3(t^2 - 4)^{1/3}} = \frac{4t}{3\sqrt[3]{t^2 - 4}}.$$

(b) Find the critical points of $h(t)$.

Solution: We find where $h'(t) = 0$ or is undefined.

- For $h'(t) = 0$, we check where the numerator $4t$ is 0, which happens at $t = 0$.
- For $h'(t)$ undefined, we check where the denominator is 0: this happens when $3(t^2 - 4)^{1/3} = 0$, so when $t^2 - 4 = 0$. Solving for t, $t = 2$ or $t = -2$.

Hence, the critical points of $h(t)$ are -2, 0, and 2.

(c) Find the intervals on which $h(t)$ is increasing and on which $h(t)$ is decreasing.

Solution: The function $h(t)$ is defined for all t, so its domain is the entire real line. The critical points divide the line into intervals on which $h'(t)$ has a single sign:

h' sign		U		0		U	
t		-2		0		2	

We evaluate $h'(t)$ at points in these intervals and record the sign:

- On $(-\infty, -2)$, we check $t = -3$: $h'(-3) = \frac{4(-3)}{3\sqrt[3]{5}}$. Since the numerator is negative and the denominator positive, $h'(-3)$ is negative.
- On $(-2, 0)$, we check $t = -1$: $h'(-1) = \frac{4(-1)}{3\sqrt[3]{-3}}$. Since the numerator and denominator are both negative, $h'(-1)$ is positive.
- On $(0, 2)$, we check $t = 3$: $h'(1) = \frac{4(1)}{3\sqrt[3]{-3}}$. Since the numerator is positive and the denominator negative, $h'(1)$ is negative.
- On $(2, \infty)$, we check $t = 3$: $h'(3) = \frac{4(3)}{3\sqrt[3]{5}}$. Since the numerator and denominator are both positive, $h'(3)$ is positive.

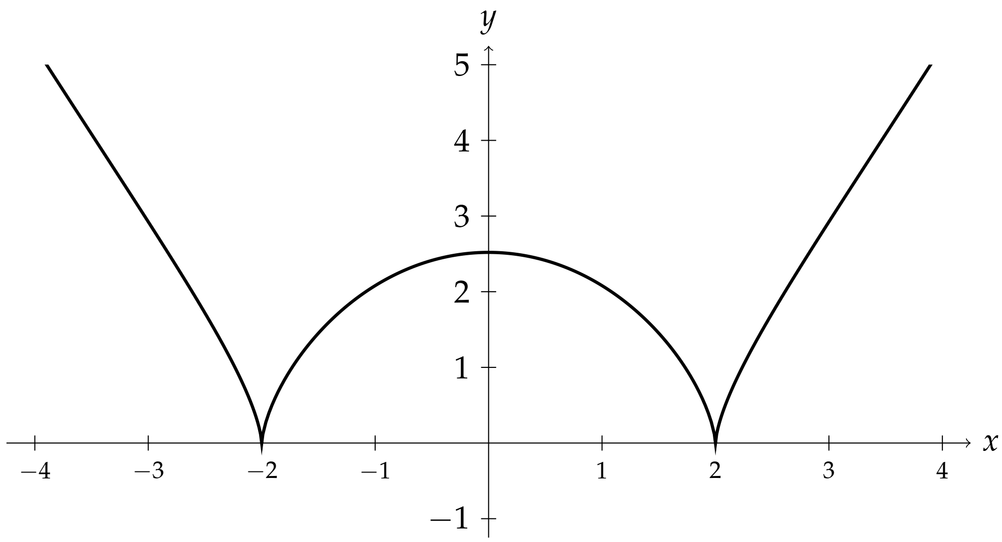
We show these signs and the corresponding $h(t)$ behavior:

$h(t)$	dec	CP	inc	CP	dec	CP	inc
h' sign	$-$	U	$+$	0	$-$	U	$+$
t		-2		0		2	

(d) Characterize each critical point as a local minimum, local maximum, or neither. Justify your answers.

Solution: At $t = 2$ and at $t = -2$, $h'(t)$ changes sign from $-$ to $+$, so $h(t)$ has local minima here. At $t = 0$, $h'(t)$ changes sign from $+$ to $-$, so $h(t)$ has a local maximum here.

Note: although the problem does not ask us to graph the function, we understand it much better if we do:



4. Below are the values of $g(t)$ for certain values of t.

t	1	3	5	7	9	11	13
$g(t)$	0.7	1.9	2.7	3.1	2.9	1.5	−0.3

(a) Estimate $g'(3)$ and $g'(11)$. Explain your estimates.

Solution: A balanced way to estimate $g'(3)$ is to take the t values immediately to the left and to the right of $t = 3$, so that $t = 3$ is centered on that interval, and compute the average rate of change over that interval. In this case, this yields

$$g'(3) \approx \frac{g(5) - g(1)}{5 - 1} = \frac{2.7 - 0.7}{4} = 0.5.$$

Similarly, we estimate that $g'(11)$ is given by

$$g''(11) \approx \frac{g(13) - g(9)}{13 - 9} = \frac{-0.3 - 2.9}{4} = \frac{-3.2}{4} = -0.8.$$

(b) Do you expect $g''(t)$ to be positive or negative on this interval? Explain.

Solution: Since our estimates of the derivative $g'(t)$ decrease from $t = 3$ to $t = 11$, we expect $g''(t)$ to be negative.

5. Our favorite budget steel mill, Bethlehem Steel, has made some changes to its steel prices. The cost in dollars of x tons of steel is now given by the function

$$C(x) = 2000 + 800x - 6x^2 + 0.05x^3.$$

(a) Find $C'(x)$.
Solution: We find that $C'(x) = 800 - 12x + 0.15x^2$.

(b) Evaluate $C(100)$ and $C'(100)$. Interpret your results, and include units.
Solution: At $x = 100$,

$$\begin{aligned} C(100) &= 200 + 800(100) - 6(100)^2 + 0.05(100)^3 \\ &= 200 + 80{,}000 - 60{,}000 + 50{,}000 = 72{,}000, \\ C'(100) &= 800 - 12(100) + 0.15(10000) = 800 - 1200 + 1500 = 1100. \end{aligned}$$

Hence, to buy 100 tons of steel, the cost is 72,000 dollars, and the cost is increasing at a rate of 1100 dollars per ton.

(c) Find an equation of the tangent line to $C(x)$ at $x = 100$.
Solution: At $x = 100$, we use the point-slope formula to find the equation of the tangent line:

$$y = C'(100)(x - 100) + C(100) = 1100(x - 100) + 72{,}000.$$

If we write this equation in slope-intercept form, $y = 1100x - 38{,}000$.

(d) Estimate $C(102)$.
Solution: We use the tangent line to estimate $C(102)$, since it provides the best linear approximation to $C(x)$ near $x = 100$. Then

$$C(102) \approx 1100(102 - 100) + 72{,}000 = 2200 + 72{,}000 = 74{,}200.$$

6. Below are values of three functions, $r(x)$, $s(x)$, and $t(x)$, and their derivatives at different values of x.

x	$r(x)$	$s(x)$	$t(x)$	$r'(x)$	$s'(x)$	$t'(x)$
3	4	0	1	2	4	3
4	2	3	3	−2	6	2
5	3	4	4	−4	7	0

(a) Let $H(x) = r(s(x))$. Find $H'(4)$.
Solution: Since $H'(x) = r'(s(x))s'(x)$ by the chain rule,

$$H'(4) = r'(s(4))s'(4) = r'(3)s'(4) = (2)(6) = 12.$$

(b) Let $L(x) = \ln(t(x))$. Find $L'(3)$.
Solution: Since $L'(x) = \frac{t'(x)}{t(x)}$ by the chain rule and the derivative of $\ln x$,

$$L'(3) = \frac{t'(3)}{t(3)} = \frac{3}{1} = 3.$$

(c) Let $P(x) = r(x)s(x)$. Find $P'(5)$.
Solution: By the product rule, $P'(x) = r'(x)s(x) + r(x)s'(x)$. Then

$$P'(5) = r'(5)s(5) + r(5)s'(5) = (-4)(4) + (3)(7) = 5.$$

(d) Let $Q(x) = \dfrac{r(x)}{t(x)}$. Find $Q'(3)$.
Solution: By the quotient rule, $Q'(x) = \dfrac{r'(x)t(x) - r(x)t'(x)}{t(x)^2}$, so

$$Q'(3) = \frac{r'(3)t(3) - r(3)t'(3)}{t(3)^2} = \frac{2(1) - 4(3)}{1} = -10.$$

(e) Let $V(x) = s(x)e^{r(x)}$. Find $V'(4)$.
Solution: By the product rule and the chain rule,

$$V'(x) = s'(x)e^{r(x)} + s(x)e^{r(x)}r'(x) = (s'(x) + s(x)r'(x))e^{r(x)}.$$

Then $V'(4) = (s'(4) + s(4)r'(4))e^{r(4)} = (6 + 3(-2))e^2 = 0$.

7. On a hot summer's day, we launch a water balloon into the air from the roof of a building. The vertical position of the balloon is given by $y(t) = 35 + 30t - 5t^2$, in meters, where t is the time in seconds since the balloon was launched.

(a) What are the balloon's vertical velocity $v(t)$ and acceleration $a(t)$? Include units.
Solution: The vertical velocity is the derivative of the height function $y(t)$, so $v(t) = y'(t) = 30 - 10t$, in units of m/s. The acceleration is the derivative of velocity, so it is $a(t) = v'(t) = -10$, in units of m/s^2.

(b) How high up does the balloon go? At what time does the balloon reach its peak?

Solution: We first find when the balloon reaches its peak. This happens when the velocity $v(t)$ is 0, as that indicates a critical point of the height function $y'(t)$. Hence, $30 - 10t = 0$, so $t = 3$. At this time, $y(3) = 35 + 30(3) - 5(3)^2 = 80$.

(c) How long does it take the balloon to hit the ground?

Solution: When the balloon hits the ground, its height is 0, so $y(t) = 35 + 30t - 5t^2 = 0$. Factoring out and dividing by a -5, $t^2 - 6t - 7 = 0$, so $(t-7)(t+1) = 0$, and $t = -1$ or $t = 7$. The solution $t = -1$ does not make sense, so we conclude that the balloon hits the ground at $t = 7$, or 7 seconds after the launch.

(d) What is the vertical velocity of the balloon when it hits the ground?

Solution: We evaluate $v(t)$ at $t = 7$: $v(7) = 30 - 10(7) = -40$. Hence, the balloon is traveling downward at 40 m/s at the time of impact.

8. When a 200-milligram dose of the drug pretendozole ingested, the function

$$C(t) = \frac{60t}{t^3 + 16}$$

describes its concentration in the bloodstream t hours later, in mg/l.

(a) Find $C'(t)$. What are the units of this quantity?

Solution: We use the quotient rule to compute $C'(t)$, with $f(t) = 60t$ and $g(t) = t^3 + 16$. Then $f'(t) = 60$ and $g'(t) = 3t^2$, so

$$C'(t) = \frac{60(t^3 + 16) - 60t(3t^2)}{(t^3 + 16)^2} = \frac{60(16 - 2t^3)}{(t^3 + 16)^2} = \frac{120(8 - t^3)}{(t^3 + 16)^2}.$$

This quantity is in units of mg/l $\cdot$ hr.

(b) Evaluate $C'(1)$ and $C'(3)$. What do these values tell you about how $C(t)$ is changing?

Solution: At $t = 1$, $C'(1) = \frac{120(8-1)}{(1+16)^2} = \frac{120(7)}{17^2} = \frac{840}{17^2}$, and at $t = 3$, $C'(3) = \frac{120(8-27)}{(27+16)^2} = \frac{120(-19)}{43^2} = \frac{-2280}{43^2}$. These values indicate that $C(t)$ is increasing around $t = 1$ and decreasing around $t = 3$.

(c) Find the time t when the maximum concentration occurs. What is the concentration at that maximum?

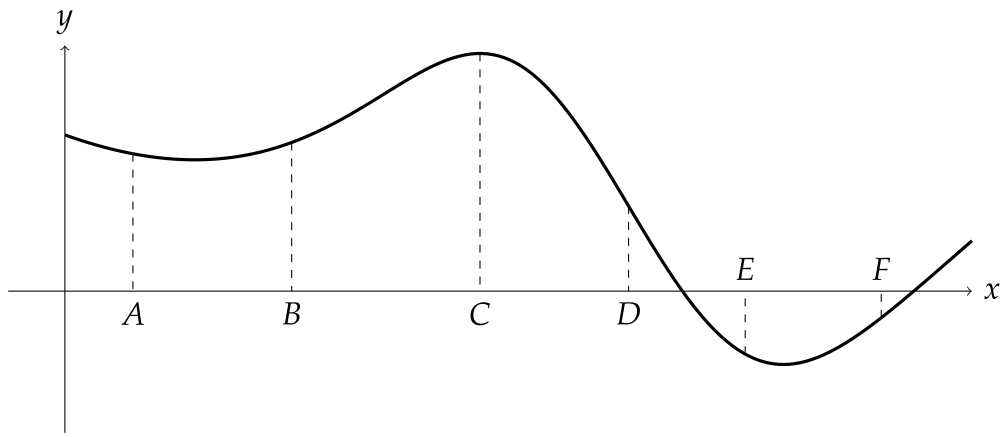
Solution: We compute the critical points of $C(t)$ for $t \geq 0$. We first look for critical points where $C'(t) = 0$. Since

$$C'(t) = \frac{120(8 - t^3)}{(t^3 + 16)^2},$$

these occur only when the numerator $120(8-t^3) = 0$, so $t^3 = 8$, and thus $t = 2$. We also note that for $t \geq 0$, $t^3 + 16$ is strictly positive, so we get no undefined-case critical points from the denominator being 0.

From part (b), $C(t)$ is increasing to the left of $t = 2$ and decreasing to the right, so $C(t)$ has a local maximum at $t = 2$ by the first derivative test. Furthermore, $C(t)$ is increasiing on $[0, 2)$ and decreasing on $(2, \infty)$, so $C(t)$ has a global maximum at $t = 2$. There, $C(2) = \frac{60(2)}{8+16} = \frac{120}{24} = 5$ mg/l.

9. Below is the graph of a function $f(x)$, labeled with points A through F.



At which of the labeled points is

(a) $f(x)$ greatest?

Solution: $f(x)$ has the greatest value at C, since the graph is highest there.

(b) $f'(x)$ greatest?

Solution: We look for where $f(x)$ has the steepest upward slope. B and F both have positive slopes, but the steepest slope occurs at F.

(c) $f(x)$ smallest?

Solution: Looking for the lowest height on the graph, we observe it at E.

(d) $f'(x)$ smallest?

Solution: This is where $f(x)$ has the steepest downward slope, which happens at D.

(e) $f'(x) = 0$?

Solution: We look for where $f(x)$ has a horizontal tangent line, and we observe that only C has such a tangent.

(f) $f''(x) = 0$?

Solution: We look for where $f(x)$ changes concavities. This occurs at D, as $f(x)$ changes from negative concavity to positive.