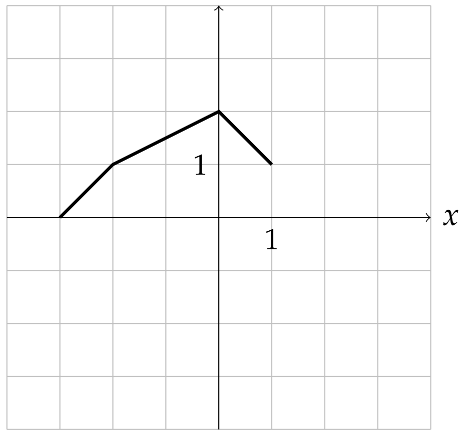
Solutions to Midterm #1 Practice Problems

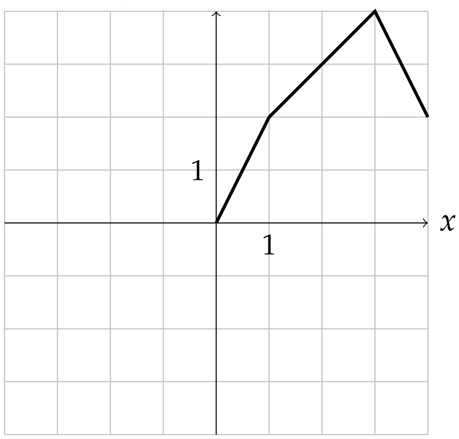
1. Below is the graph of a function $y = r(x)$.

$y = r(x)$

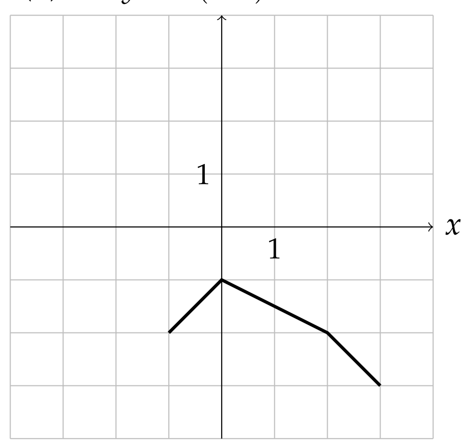


Sketch graphs of the following functions:

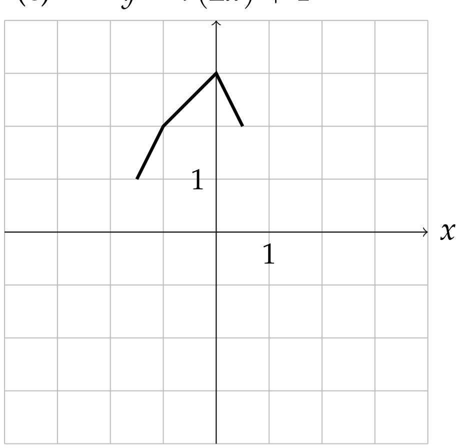
(a) $y = 2r(x-3)$



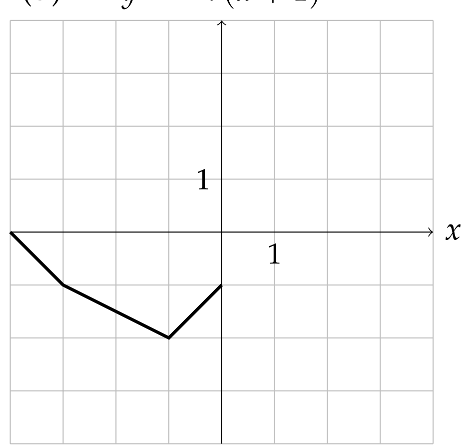
(b) $y = r(-x) - 3$



(c) $y = r(2x) + 1$



(d) $y = -r(x+1)$



2. The equation $7x - 11y = 16$ describes a line in the xy-plane.

(a) Find the slope of the line.

Solution: Solve for y: $11y = 7x - 16$, so $y = \frac{7}{11}x - \frac{16}{11}$. Hence, the slope is $\frac{7}{11}$.

(b) What is the y-intercept of the line?

Solution: From the equation above in slope-intercept form, the y-intercept is $-\frac{16}{11}$.

(c) What is the x-intercept of the line?

Solution: Set $y = 0$ in the equation of the line. Then $7x = 16$, so $x = \frac{16}{7}$ is the x-intercept.

(d) Is the point $(7, 3)$ on this line?

Solution: We check that $x = 7$ and $y = 3$ satisfies the equation:

$$7(7) - 11(3) = 49 - 33 = 16$$

It does, so the point is on the line.

3. QwikWidgets, Inc., manufactures widgets (as you might expect). On a given production run, they can make 10,000 two-inch stainless-steel widgets for \$1200 and 50,000 for \$4400. Assume that their costs change linearly with their production size.

(a) Once their widget-producing machines are up and running, how much does it cost to produce an extra widget? (This is called the *marginal cost* of production.)

Solution: We compute the average rate of change in the cost of widgets:

$$m = \frac{\Delta C}{\Delta w} = \frac{4400 - 1200}{50000 - 10000} = \frac{3200}{40000} = \frac{8}{100} = 0.08.$$

Hence, each widget costs \$0.08 to make.

(b) How much would it cost QwikWidgets only to turn on the widget-making machines, without actually making any widgets at all?

Solution: The 10,000 widgets cost $\$0.08 \times 10{,}000 = \$800$, so the remaining cost, $1200 - 800 = \$400$, is the cost of starting up the factory.

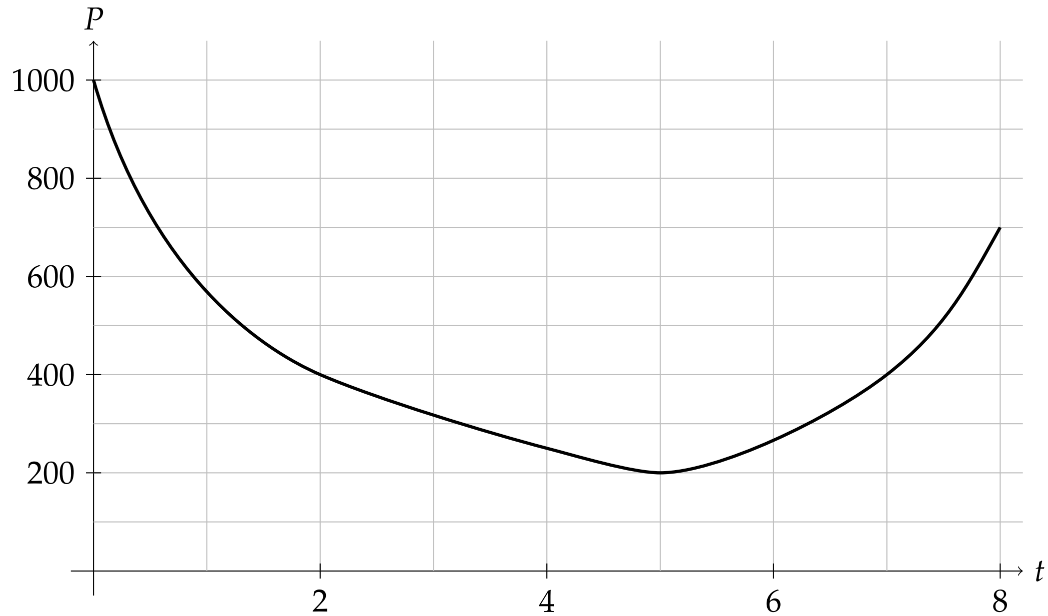
(c) Write a formula for the cost of making w widgets on a given production run.

Solution: The cost is $C(w) = 0.08w + 400$, in dollars.

(d) How much would it cost to manufacture 100,000 widgets?

Solution: The cost would be $C(100{,}000) = 0.08(100{,}000) + 400 = 8400$ dollars.

4. Below is a graph of the population $P(t)$ of wolves in a forest t years after the year 2000.



(a) Over which time intervals is the graph increasing? decreasing? concave up? concave down?

Solution: The graph is increasing on $[5, 8]$ and decreasing on $[0, 5]$. It is concave up on $[0, 8]$ and is concave down nowhere.

(b) What is the average rate of change of the population from 2000 to 2002?

Solution: $P(2000) = 1000$ and $P(2002) = 400$, so the average rate of change is $m = \frac{400 - 1000}{2002 - 2000} = \frac{-600}{2} = -300$ wolves per year.

(c) What is the percentage change in the population from 2007 to 2008?

Solution: $P(2007) = 400$ and $P(2008) = 700$, so the relative change is $\frac{700 - 400}{400} = \frac{300}{400} = \frac{3}{4}$. The corresponding percentage change is 75%.

5. Savvy Sally invests \$5000 in a mutual fund and receives a steady interest rate of 8%.

(a) If the interest is compounded annually, write a function $V(t)$ giving the value of Sally's investment after t years.
Solution: The value function is $V(t) = 5000(1.08)^t$.

(b) What is the formula for $V(t)$ if the interest is instead compounded quarterly?
Solution: The quarterly interest rate is $8/4 = 2\%$, so the formula in this case is $V(t) = 5000(1.02)^{4t}$.

(c) What is the formula for $V(t)$ if the interest is instead compounded continuously?
Solution: The value under continuous compounding is $V(t) = 5000e^{0.08t}$.

6. In each equation, solve for x symbolically.

(a) $25 = 3^x$
Solution: The x is in the exponent, so we use a logarithm to extract it: $\log_3(25) = \log_3(3^x) = x$, so $x = \log_3(25)$.

(b) $x^5 = 79$
Solution: Here, the x is in the base, so we take 5th-roots: $x = \sqrt[5]{79}$.

(c) $3 = \ln(2x + 5)$
Solution: We exponentiate both sides of the equation at the base e: $e^3 = e^{\ln(2x+5)} = 2x + 5$. We then isolate x:

$$x = \frac{e^3 - 5}{2}.$$

7. Define functions $f(x) = 2x - 3$ and $g(x) = x^2 - x$. Find formulas for:

(a) $f(g(x))$
Solution: $f(g(x)) = f(x^2 - x) = 2(x^2 - x) - 3 = 2x^2 - 2x - 3$.

(b) $g(f(x))$
Solution: $g(f(x)) = g(2x - 3) = (2x - 3)^2 - (2x - 3) = 4x^2 - 14x + 12$.

(c) $g(g(x))$
Solution: $g(g(x)) = g(x^2 - x) = (x^2 - x)^2 - (x^2 - x) = x^4 - 2x^3 + x$.

(d) $f(f(x))$
Solution: $f(f(x)) = f(2x - 3) = 2(2x - 3) - 3 = 4x - 9$.

8. Foolish Frank invests in penny stocks. The value of his investment is given by

$$V(t) = 4500e^{-0.12t}$$

where t, in years, is the age of the investment.

(a) How much did Frank invest initially?

Solution: Frank initially invested \$4500.

(b) What is the continuous growth rate of Frank's investment?

Solution: The continuous growth rate is -0.12.

(c) Find the time t when Frank's investment reaches \$1000. You do not need a decimal value for this t, but it should be an expression you could evaluate on a calculator.

Solution: We set $V(t) = 1000$, so $1000 = 4500e^{-0.12t}$. Then $e^{-0.12t} = \dfrac{1000}{4500}$, so $-0.12t = \ln\left(\dfrac{1000}{4500}\right)$. Hence,

$$t = \frac{\ln \frac{1000}{4500}}{-0.12}.$$

(Depending on which logarithm properties are used, other forms of this answer are also acceptable.)

9. Below are tables of values for functions $h(x)$, $j(x)$, and $k(x)$ at different values of x.

x	-2	-1	0	1	2
$h(x)$	48	24	12	6	3
$j(x)$	48	40	31	21	10
$k(x)$	48	41	34	27	20

(a) Is $h(x)$ linear, exponential, or neither? Write a formula for $h(x)$, if possible.

Solution: Checking the successive average rates of change, we obtain -24 and -12 as the first two rates, so the function is not linear. Checking the successive ratios between values, we obtain $\frac{1}{2}$ each time, so the function could be exponential. A formula for h is $h(x) = 12\left(\dfrac{1}{2}\right)^x$.

(b) Is $j(x)$ linear, exponential, or neither? Write a formula for $j(x)$, if possible.

Solution: The successive average rates of change start out as -8 and -9, which are not the same, so the function is not linear. The successive ratios between values start out as $\dfrac{40}{48} = \dfrac{5}{6}$ and $\dfrac{31}{40}$, which are different, so the function is also not exponential.

(c) Is $k(x)$ linear, exponential, or neither? Write a formula for $k(x)$, if possible.
Solution: The average rates of change between adjacent values are constant at -7, so the function could be linear. A formula for k is $k(x) = 34 - 7x$.

10. You acquire a 50-gram sample of iodine-131, which has a half-life of 8 days.

(a) Write a function $f(t)$ that represents the amount of iodine left after t days.
Solution: One formula for f is $f(t) = 50(2)^{-t/8}$. Putting this exponential into the continuous-growth-rate form, we also have $f(t) = 50e^{-\frac{\ln 2}{8}t}$.

(b) How long will it take for the sample to decay to 1 gram of iodine-131? Write an expression you could evaluate on your calculator.
Solution: If 1 gram remains, then $f(t) = 1$, so $50(2)^{-t/8} = 1$. Then $2^{-t/8} = \frac{1}{50}$. Taking logs, $-t/8 = \log_2 \left(\frac{1}{50}\right) = -\log_2 50$, after simplifying the logarithm. Finally,

$$t = 8 \log_2 50.$$