# Homework #8 Solutions

Problems

Bolded problems are worth 2 points.

- Section 4.1: 2, **8**, **12**, 16, **18**, 22

- Section 4.2: 4, **10**, **14**, 16, **22**, 30. On 14 and 16, use the first or second derivative tests to characterize each critical point, and confirm your answer with a graph.

- **Extra Problem**

4.1.2. Indicate the critical points of the function f. How many critical points are there? Identify each critical point as a local maximum, a local minimum, or neither.

*Solution:* We mark the locations of the critical points on the graph:
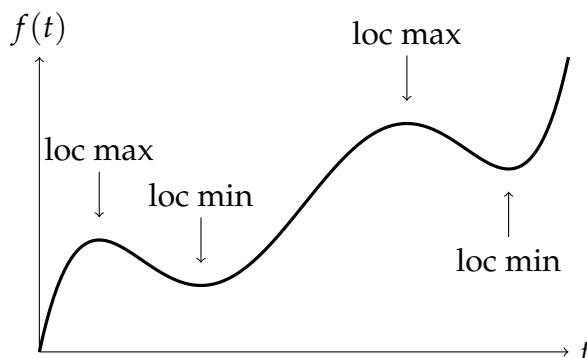
We observe a total of 4 critical points of $f(t)$. Two critical points occur at the top of the two peaks, and therefore are local maxima, and two occur at the bottom of the two valleys, and therefore are local minima.   ∎

4.1.8. Below is shown a graph of a derivative function f'. Indicate on a sketch the x-values that are critical points of the function f itself. Identify each point as a local maximum, a local minimum, or neither.

Solution: From the graph, we see that the two x-intercepts of the graph correspond to critical points of $f(x)$, since $f'(x) = 0$ at each x-intercept. For the left critical point, $f'(x)$ is positive to its left and negative to its right, so by the First Derivative Test, $f(x)$ has a local maximum there. Converserly, the right critical point has $f'(x)$ negative to its left and $f'(x)$ positive to its right, so $f(x)$ has a local minimum there.   ∎

4.1.12. Graph the function $f(x) = 3x^5 - 5x^3$. Describe in words the interesting features of the graph, including the location of the critical points and where the function is monotonic (that is, increasing or decreasing). Then use the derivative and algebra to explain the shape of the graph.

Solution: We produce the following graph of $f(x) = 3x^5 - 5x^3$:
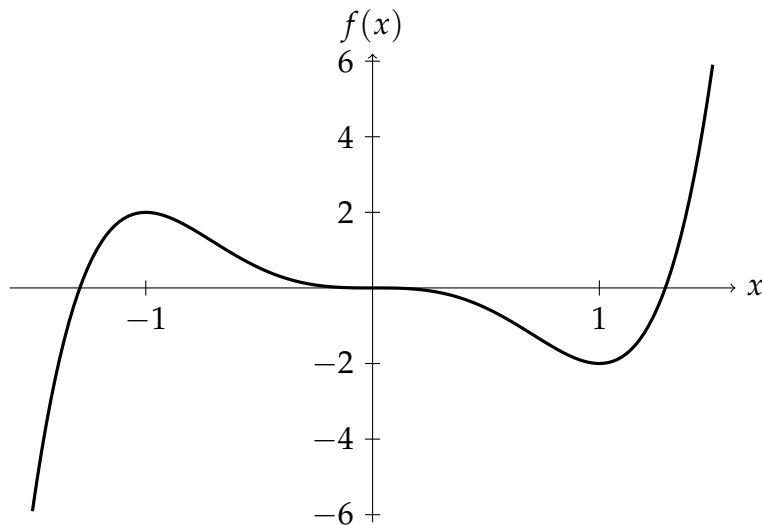
We observe that the function has a horizontal slope at about $x = 1$, $x = 0$, and $x = -1$, and therefore has critical points at these points. At $x = -1$, we see a local maximum; at $x = 1$, a local minimum; and at $x = 0$, neither type of extremum. The function is increasing for $x < -1$ and for $x > 1$, and is decreasing for $-1 < x < 1$.

We recover these results algebrically as well: taking the derivative of $f(x)$ gives

$$f'(x) = 3(5x^4) - 5(3x^2) = 15x^4 - 15x^2 = 15x^2(x^2 - 1) = 15x^2(x - 1)(x + 1).$$

Solving $f'(x) = 0$, we see that $x = 0$, $x = 1$, and $x = -1$, exactly the critical points we observed on the graph.

We also check the sign of $f'(x)$ to tell us where $f(x)$ is monotonic: since the $15x^2$ factor is always positive or 0, the $x^2 - 1$ determines the sign of $f'(x)$. It is negative when $-1 < x < 1$ and positive when $x > 1$ or $x < -1$, which exactly matches what we observed from the graph. Finally, using the sign of the derivative in the first derivative test confirms the above characterization of the critical points. ∎

4.1.16. Below is a graph of f'. For what values of x does f have a local maximum? A local minimum?

$f'(x)$

x

1 3 5
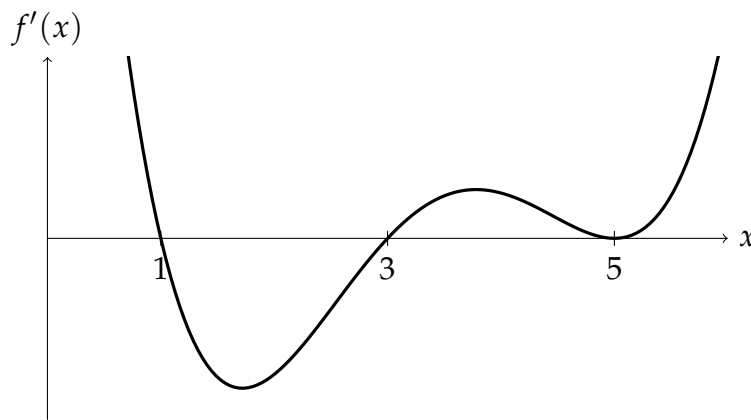
Solution: From the graph, we observe that $f'(x) = 0$ at $x = 1$, $x = 3$, and $x = 5$, so f has critical points at these three x-values. We use the sign of the derivative around these points and the first derivative test to characterize the critical points:

- $f'(x)$ is positive to the left of $x = 1$ and negative to the right, so $f(x)$ has a local maximum there.

- $f'(x)$ is negative to the left of $x = 3$ and positive to the right, so $f(x)$ has a local maximum there.

- $f'(x)$ is positive to the left of $x = 5$ and to the right, so $f(x)$ does not have a local extremum there. ∎

4.1.18. The derivative of $f(t)$ is given by $f'(t) = t^3 - 6t^2 + 8t$ for $0 \leq t \leq 5$. Graph $f'(t)$, and describe how the function changes over the interval $t = 0$ to $t = 5$. When is $f(t)$ increasing and when is it decreasing? Where does $f(t)$ have a local maximum and where does it have a local minimum?

Solution: The graph of $f'(t) = t^3 - 6t^2 + 8t$ is shown below:

$f'(t)$

t

1 2 3 4 5
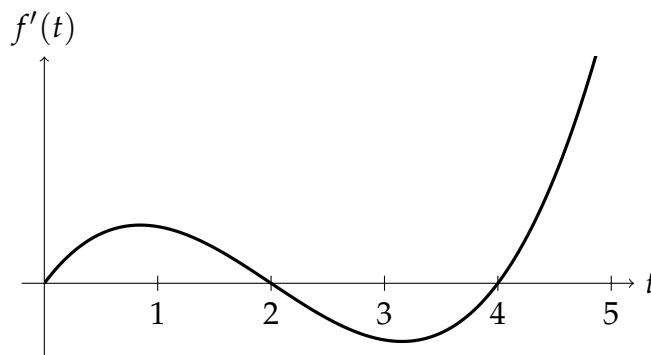
We see that $f'(t)$ is positive on the intervals $(0, 2)$ and $(4, 5]$, and is negative on the interval $(2, 4)$. Therefore, $f(t)$ is increasing from $t = 0$ to $t = 2$, decreasing from from $t = 2$ to

$t = 4$, and increasing again from $t = 4$ to $t = 5$. $f'(t) = 0$ at $t = 0$, $t = 2$, and $t = 4$, so $f(t)$ has critical points there. By the first derivative test, $f(t)$ has local minima at $t = 0$ and $t = 4$, and a local maximum at $t = 2$. ■

4.1.22. The function $f(x) = x^4 - 4x^3 + 8x$ has a critical point at $x = 1$. Use the second derivative test to identify it as a local maximum or a local minimum.

Solution: We compute the second derivative of $f(x)$: $f'(x) = 4x^3 - 12x^2 + 8$, so $f''(x) = 12x^2 - 24x$. At $x = 1$, $f''(1) = 12 - 24 = -12$. Since $f''(1)$ is negative, the critical point $x = 1$ is a local maximum for $f(x)$. ■

4.2.4. On the given graph, indicate the approximate locations of all inflection points. How many inflection points are there?


Solution: We indicate the inflection points on the graph, at the locations where the concavity changes:
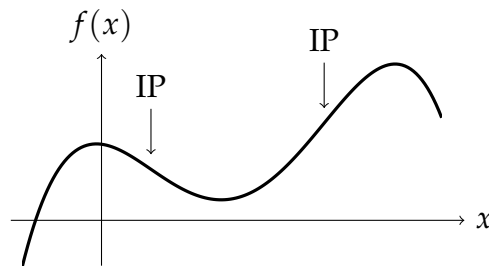
Since the concavity of $f(x)$ changes from negative to positive and back to negative, we observe two inflection points. ■

4.2.10. For $f(x) = x^3 - 18x^2 - 10x + 6$, find the inflection point algebraically. Graph the function with a calculator and confirm your answer.

Solution: We find where $f''(x) = 0$: first, $f'(x) = 3x^2 - 36x - 10$, so $f''(x) = 6x - 36$. Setting $f''(x) = 0$, we have $6x - 36 = 0$, so $x = 6$. Therefore, we look for an inflection point here. Since $f''(x)$ is negative for $x < 6$ and positive for $x > 6$, the concavity of $f(x)$ does change here, so $f(x)$ does have an inflection point. We graph the function around $x = 6$ to confirm:
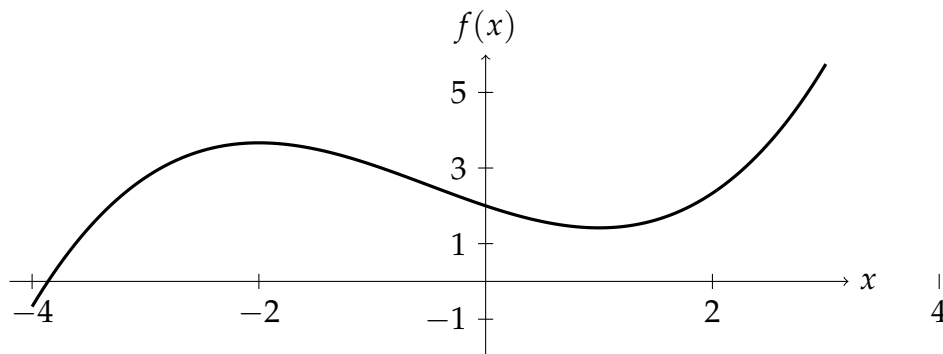

■
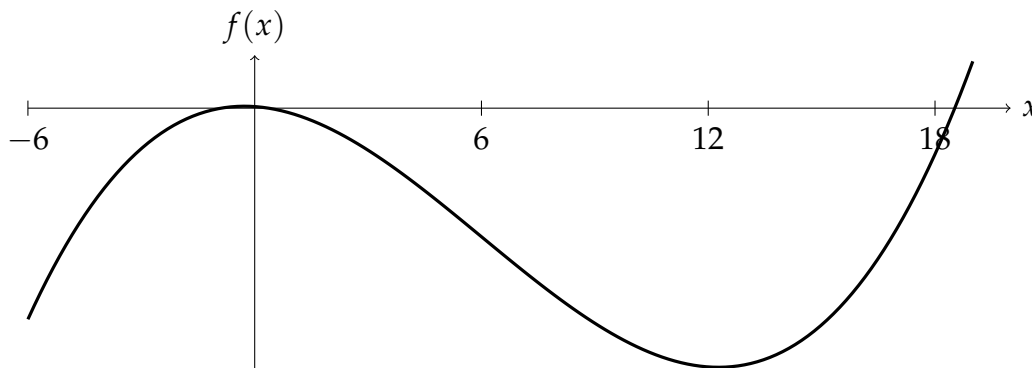
4.2.14. Let $f(x) = \dfrac{x^3}{6} + \dfrac{x^2}{4} - x + 2$. Use the first derivative to find all critical points and the second derivative to find all inflection points. Characterize the critical points using a derivative test, and confirm your answer with a graph.

Solution: We compute the first derivative of $f(x)$: $f'(x) = \dfrac{3x^2}{6} + \dfrac{2x}{4} - 1 = \dfrac{1}{2}x^2 + \dfrac{1}{2}x - 1$.
To find the critical points, we set $f'(x) = 0$. Then $\frac{1}{2}x^2 + \frac{1}{2}x - 1 = 0$, so $x^2 + x - 2 = 0$. This factors as $(x + 2)(x - 1) = 0$, so $x = -2$ and $x = 1$ are solutions and are therefore critical points of $f(x)$.

The second derivative is $f''(x) = x + \frac{1}{2}$. Setting this equal to 0 and solving, we see that $x = -\frac{1}{2}$. Furthermore, $f''(x)$ changes from negative to positive here, so $x = -\frac{1}{2}$ is an inflection point of $f(x)$.

To characterize the critical points at $x = -2$ and $x = 1$, we use the second derivative test. At $x = -2$, $f''(x) = -2 + \frac{1}{2} = -\frac{3}{2}$. Since $f''(x)$ is negative here, $f(x)$ has a local maximum here. At $x = 1$, $f''(1) = 1 + \frac{1}{2}$, which is positive, so $f(x)$ has a local minimum here. We check these results with a graph:


4.2.16. Let $f(x) = 3x^4 - 4x^3 + 6$. Use the first derivative to find all critical points and the second derivative to find all inflection points. Characterize the critical points using a derivative test, and confirm your answer with a graph.

Solution: We compute the first derivative of $f(x)$: $f'(x) = 12x^3 - 12x^2$. We then solve $f'(x) = 0$: $12x^3 - 12x^2 = 12x^2(x - 1) = 0$, so either $x = 0$ or $x = 1$. Hence, these are the critical points of $f(x)$.

We compute $f''(x) = 36x^2 - 24x$. To find the inflection points, we set $f''(x) = 0$, so $36x^2 - 24x = 12x(3x - 2) = 0$. The solutions are therefore $x = 0$ and $x = 2/3$. We will use the graph below to check that these are both inflection points.

To characterize the critical points, we first evaluate $f''(x)$ at each of them. At $x = 1$, $f''(x) = 36 - 24 = 12$, so we observe a local minimum of $f(x)$ there. At $x = 0$, however, $f''(x) = 0$, so the second derivative test is inconclusive, and we must fall back to the first derivative test.

Since $f(x)$ has a local minimum at $x = 1$, $f(x)$ must be decreasing from $x = 0$ to $x = 1$ (the region between the two critical points). Therefore, we need only determine whether $f(x)$ is increasing or decreasing to the left of $x = 0$. We look at $f'(x)$ at $x = -1$: $f'(-1) = 12(-1)^2(-1-1) = -24$, so $f(x)$ is decreasing to the left of $x = 0$. Since $f(x)$ is decreasing on both sides of $x = 0$, it has neither a local minimum nor a local maximum there. We confirm this behavior with a graph of $f(x)$:
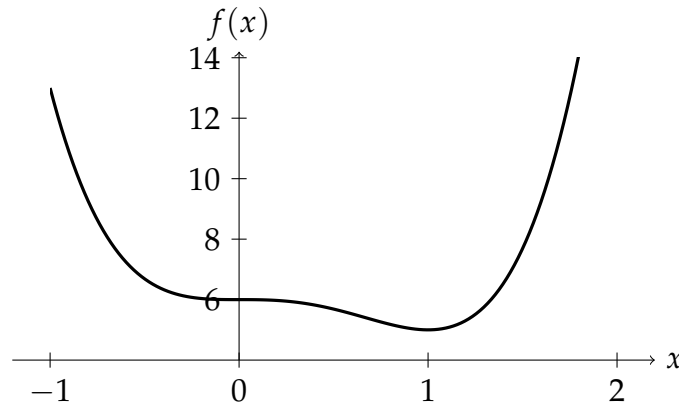
We also see changes in concavity at both $x = 0$ and $x = 2/3$, so they are both indeed inflection points. ∎

4.2.22. Let $f(x) = x^4 - 2ax^2 + b$, where a and b are positive constants.

(a) Find all critical points and inflection points of $f(x)$.

(b) Find values of the parameters a and b if f has a critical point at the point $(2, 5)$.

(c) If there is a critical point at $(2, 5)$, where are the inflection points?

Solution (a): We first find $f'(x) = 4x^3 - 4ax$. Setting this to 0, we have $4x^3 - 4ax = 4x(x^2 - a) = 0$. Therefore, either $x = 0$ or $x^2 = a$, in which case $x = \sqrt{a}$ or $x = -\sqrt{a}$. These three x-values give the locations of the critical points of x in terms of the parameter a.

For the inflection points, we compute $f''(x) = 12x^2 - 4a$ and set it to 0. Then $12x^2 - 4a = 0$, so $12x^2 = 4a$, and $x^2 = a/3$. The solutions are therefore $x = \sqrt{a/3}$ and $x = -\sqrt{a/3}$. Since $f''(0) = -4a$, which is negative, but since $f''(x)$ is positive for large values of x, the sign of $f''(x)$ actually does change at these points, so they are inflection points. ∎

Solution (b): If $f(x)$ has a critical point at $(2, 5)$, then $x = 2$ must match one of the critical point values we computed in part (a). Of those, the only positive value was $\sqrt{a}$, so $\sqrt{a} = 2$, and $a = 4$.

To find b, we plug $a = 4$ into the definition of $f(x)$ and then evaluate it at $x = 2$. The output should be the y-coordinate, 5, so

$$5 = (2)^4 - 2(4)(2)^2 + b = 16 - 32 + b = b - 16.$$

Thus, $b = 5 + 16 = 21$. ∎

Solution (c): From part (b), we know that for $f(x)$ to have a critical point at $(2,5)$, then $a = 4$. From part (a), the critical points of $f(x)$ are given in terms of a as $\sqrt{a/3}$ and $-\sqrt{a/3}$, so this particular $f(x)$ has inflection points at $\sqrt{4/3}$ and $-\sqrt{4/3}$.   ∎

4.2.30. If water is flowing at a constant rate (i.e., constant volume per unit time) into the vase in Figure 4.26, sketch a graph of the depth of the water against time. Mark on the graph the time at which the water reaches the corner of the vase.

Solution: Since volume inside the vase is the product of the height of the vase and the cross-sectional area, and since the volume increases at a constant rate, the rate of change of the height is inversely proportional to the cross-sectional area at that height. Therefore, as the area gets bigger going up the vase, the height will increase more slowly.

As a result, if we graph the height over time, we expect it to start out with a relatively steep positive slope and then to gradually get shallower as the height approaches the corner of the vase. Once it passes the corner, the cross-sectional area of the vase is constant. Therefore, the height will increase at a constant rate, and its graph will be a straight line after the "corner" point.

∎
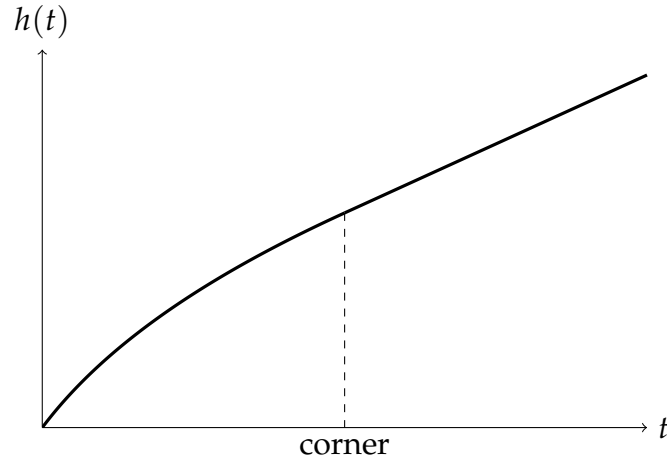
Extra Problem. Let $f(x) = x^2 - 6x + 4\ln x$. Note that the domain of this function is $x > 0$.

(a) Find $f'(x)$ and $f''(x)$.

(b) Find the critical points of $f(x)$.

(c) Find the intervals on which f is increasing and on which f is decreasing.

(d) Using the First Derivative Test, determine whether each critical point of f is a local minimum, a local maximum, or neither.

(e) Use the Second Derivative Test to check your answers to part (d). If that is not possible, explain why.

(f) Find the inflection points of $f(x)$.

Solution (a): We find $f'(x)$ and $f''(x)$:

$$f'(x) = 2x - 6 + \frac{4}{x} \qquad\qquad f''(x) = 2 - \frac{4}{x^2} \qquad\qquad \blacksquare$$

Solution (b): We set $f'(x) = 0$ and solve for x: starting with $2x - 6 + \frac{4}{x} = 0$, we multiply both sides by x to clear the denominators, and get $2x^2 - 6x + 4 = 0$. Factoring the left side, we have $2(x^2 - 3x + 2) = 2(x - 2)(x - 1) = 0$, so the solutions are $x = 1$ and $x = 2$. These are therefore the critical points of $f(x)$.
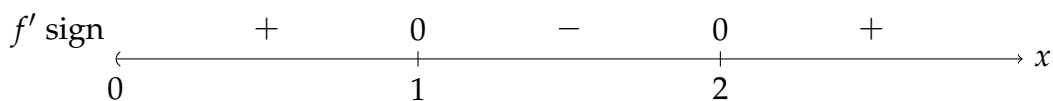
We note that $f'(x)$ is undefined at $x = 0$, but $x = 0$ is not in the domain of f, so we ignore this point. $\blacksquare$

Solution (c): The critical points $x = 1$ and $x = 2$ divide the interval $(0, \infty)$ on which $f(x)$ is defined into intervals on which $f'(x)$ has a single sign:

$$\xleftarrow{\hspace{2cm}} \underset{0}{} \underset{1}{|} \underset{2}{|} \xrightarrow{\hspace{3cm}} x$$
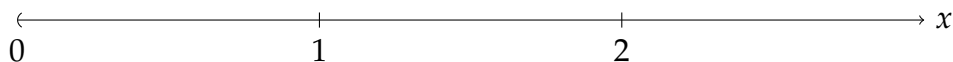
We check the sign of $f'(x)$ on each interval. At $x = 1/2$, $f'(1/2) = 1 - 6 + 8 = 3$, so $f(x)$ is increasing on $(0, 1)$. At $x = 3/2$, $f'(3/2) = 3 - 6 + \frac{8}{3} = -\frac{1}{3}$, so $f(x)$ is decreasing on $(1, 2)$. Finally, $f'(4) = 8 - 6 + 1 = 3$, so $f(x)$ is again increasing on $(2, \infty)$. $\blacksquare$

Solution (d): Using the information from part (c), we redraw the intervals above with the signs of the first derivative:

$$f' \text{ sign} \quad \overset{+}{} \quad \overset{0}{\underset{1}{|}} \quad \overset{-}{} \quad \overset{0}{\underset{2}{|}} \quad \overset{+}{} \xrightarrow{\hspace{1cm}} x$$

At $x = 1$, $f'(x)$ changes from positive to negative, so $f(x)$ has a local maximum there. At $x = 2$, $f'(x)$ changes from negative to positive, so $f(x)$ has a local maximum there. $\blacksquare$

Solution (e): We evaluate $f''(x)$ at $x = 1$ and $x = 2$. At $x = 1$, $f''(1) = 2 - 4 = -2$, which is negative, so that confirms that $f(x)$ has a local maximum at $x = 1$. At $x = 2$, $f''(2) = 2 - 1 = 1$, which is positive, so that confirms that $f(x)$ has a local minimum at $x = 2$.   ■

Solution (f): We solve $f''(x) = 0$ to find candidate locations for the inflection points. Then $2 - \dfrac{4}{x^2} = 0$, so $2 = \dfrac{4}{x^2}$, and $2x^2 = 4$. Hence, $x^2 = 2$, so $x = \sqrt{2}$ or $x = -\sqrt{2}$. Of these two solutions, only $x = \sqrt{2}$ is positive and is thus in the domain of $f(x)$. From part (e), we see that $f''(x)$ is negative to the left of $x = \sqrt{2}$ and is positive to the right, so $f''(x)$ does change sign at $x = \sqrt{2}$. Thus, $f(x)$ has an inflection point there.   ■