Homework #7 Solutions

Problems

Bolded problems are worth 2 points.

- Section 3.4: **2**, 6, 14, **16**, 24, 36, **38**, **42**
- Chapter 3 Review (pp. 159–162): **24**, 34, 36, 54, **66**
- **Extra Problem**

3.4.2. If $f(x) = x^2(x^3+5)$, find $f'(x)$ two ways: by using the product rule and by multiplying out before taking the derivative. Do you get the same result? Should you?

Solution: We first use the product rule:

$$f'(x) = (x^2)'(x^3+5) + (x^2)(x^3+5) = 2x(x^3+5) + x^2(3x^2)$$
$$= 2x^4 + 10x + 3x^4 = 5x^4 + 10x.$$

We also multiply $f(x)$ out and take the derivative: $f(x) = x^5 + 5x^2$, so $f(x) = 5x^4 + 10x$. This is the same result as with the product rule, as expected. ■

3.4.6. Find the derivative of $y = t^2(3t+1)^3$.

Solution: We use the product rule and the generalized power rule to compute this derivative:

$$y' = (t^2)'(3t+1)^3 + t^2\left((3t+1)^3\right)' = 2t(3t+1)^3 + t^2(3)(3t+1)^2(3)$$

Simplifying,

$$y' = (2t(3t+1) + 9t^2)(3t+1)^2 = (15t^2 + 2t)(3t+1)^2.$$ ■

3.4.14. Find the derivative of $f(x) = \dfrac{x^2+3}{x}$.

Solution: Rather than use the quotient rule immediately, we rewrite $f(x)$ first: $f(x) = (x^2+3)x^{-1} = x + 3x^{-1}$. Hence,

$$f'(x) = 1 + 3(-x^{-2}) = 1 - \frac{3}{x^2}.$$ ■

3.4.16. Find the derivative of $f(z) = \sqrt{z}e^{-z}$.

Solution: We use the product rule:

$$f'(z) = (\sqrt{z})'e^{-z} + \sqrt{z}(e^{-z})' = \frac{1}{2\sqrt{z}}e^{-z} + \sqrt{z}(-e^{-z}).$$

Simplifying further,

$$f'(z) = \frac{1-2z}{2\sqrt{z}}e^{-z} = \frac{1-2z}{\sqrt{z}e^{z}}.$$

■

3.4.24. Find the derivative of $w = \dfrac{3z}{1+2z}$.

Solution: In this case, we use the quotient rule, with $f(z) = 3z$ and $g(z) = 1+2z$. Then $f'(z) = 3$ and $g'(z) = 2$, so

$$w' = \frac{3(1+2z) - (3z)(2)}{(1+2z)^2} = \frac{3+6z-6z}{(1+2z)^2} = \frac{3}{(1+2z)^2}.$$

■

3.4.36. Find the equation of the tangent line to the graph of $f(x) = \dfrac{2x-5}{x+1}$ at the point where $x = 0$.

Solution: At $x = 0$, $f(x) = f(0) = \frac{0-5}{0+1} = -5$, so the tangent line goes through the point $(0, -5)$. We compute the derivative to be

$$f'(x) = \frac{2(x+1) - (2x-5)(1)}{(x+1)^2} = \frac{2x+2-2x+5}{(x+1)^2} = \frac{7}{(x+1)^2}.$$

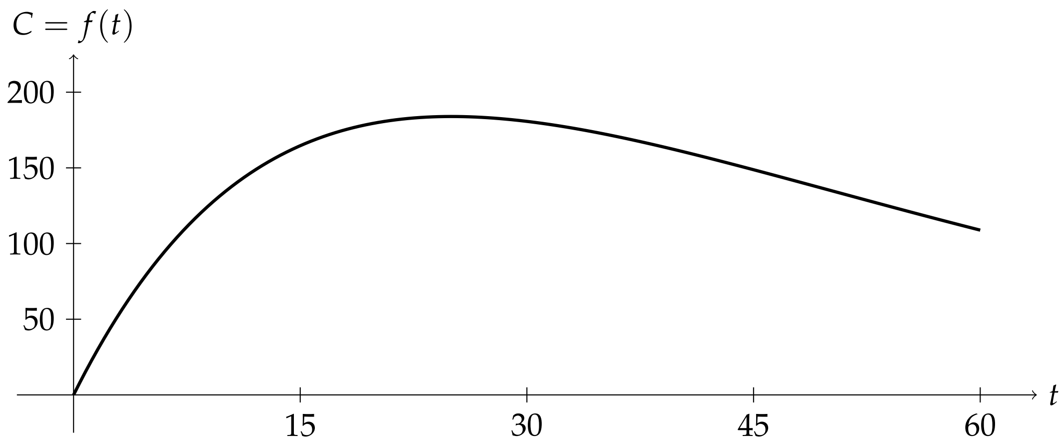
Then the slope of the tangent line is $f'(0) = \frac{7}{(0+1)^2} = 7$. Therefore, the line is $y = 7x - 5$.■

3.4.38. A drug concentration curve is given by $C = f(t) = 20te^{-0.04t}$, with C in mg/ml and t in minutes.

(a) Graph C against t. Is $f'(15)$ positive or negative? Is $f'(45)$ positive or negative? Explain.

(b) Find $f(30)$ and $f'(30)$ analytically. Interpret them in terms of the concentration of the drug in the body.

Solution (a): We produce the following graph of $f(t)$, from $t = 0$ to $t = 60$:



Hence, we can see directly that $f'(15)$ is positive and $f'(45)$ is negative. ∎

Solution (b): First, $f(30) = 20(30)e^{-0.04(30)} = 600e^{-1.2} \approx 180.7$ mg/ml, so this is the concentration of the drug in the body 30 minutes after the dose. Using the product rule, we compute $f'(t)$ to be

$$f'(t) = 20[(1)e^{-0.04t} + t(-0.04)e^{-0.04t}] = 20(1 - 0.04t)e^{-0.04t}.$$

Hence, $f'(30) = 20(1 - 1.2)e^{-1.2} = -4e^{-1.2} \approx -1.205$ mg/ml · min, so this is the rate at which the drug concentration is changing 30 minutes after taking the dose. Since it is negative, the concentration is decreasing at this time. ∎

3.4.42. The quantity demanded of a certain product, q, is given in terms of p, the price, by

$$q = 1000e^{-0.02p}.$$

(a) Write revenue, R, as a function of price.

(b) Find the rate of change of the revenue with respect to price.

(c) Find the revenue and rate of change of revenue with respect to price when the price is \$10. Interpret your answer in economic terms.

Solution (a): The revenue is $R(p) = pq(p) = 1000pe^{-0.02p}$. ∎

Solution (b): The marginal revenue, or the rate of change of the revenue, is the derivative

$$R'(p) = 1000((1)e^{-0.02p} + p(-0.02e^{-0.02p})) = 1000(1 - 0.02p)e^{-0.02p}.$$ ∎

Solution (c): When $p = 10$, $R(10) = 1000(10)p^{-0.02(10)} = 10{,}000e^{-0.2} \approx 8187.31$, and $R'(10) = 1000(1 - 0.2)e^{-0.2} = 800e^{-0.2} \approx 654.99$, in dollars per dollar. Hence, the revenue is 8187 dollars at this price, but it is increasing at a rate of about 655 dollars per dollar of price. Consequently, we expect to be able to get more revenue by raising the price. ∎

3.R.24. Find the derivative of the function $h(t) = \dfrac{t+4}{t-4}$.

Solution: Using the quotient rule,

$$h'(t) = \frac{(1)(t-4) - (t+4)(1)}{(t-4)^2} = \frac{t-4-t-4}{(t-4)^2} = -\frac{8}{(t-4)^2}.$$

Alternately, we simplify $h(t)$ first:

$$h(t) = \frac{t+4}{t-4} = \frac{(t-4)+8}{(t-4)} = 1 + 8(t-4)^{-1}.$$

Then by the chain rule,

$$h'(t) = 8(-1)(t-4)^{-2}(1) = -8(t-4)^{-2} = -\frac{8}{(t-4)^2}.$$

■

3.R.34. Find the derivative of $z = \dfrac{3t+1}{5t+2}$.

Solution: We use the quotient rule:

$$z' = \frac{3(5t+2) - (3t+1)(5)}{(5t+2)^2} = \frac{15t+6-15t-5}{(5t+2)^2} = \frac{1}{(5t+2)^2}.$$

3.R.36. Find the derivative of $h(p) = \dfrac{1+p^2}{3+2p^2}$.

Solution: We use the quotient rule:

$$h'(p) = \frac{2p(3+2p^2) - (1+p^2)(4p)}{(3+2p^2)^2} = \frac{6p+4p^3-4p-4p^3}{(3+2p^2)^2} = \frac{2p}{(3+2p^2)^2}.$$

3.R.54. Fnd the equation of the tangent line to the graph of $P(t) = t \ln t$ at $t = 2$. Graph the function $P(t)$ and the tangent line $Q(t)$ on the same axes.

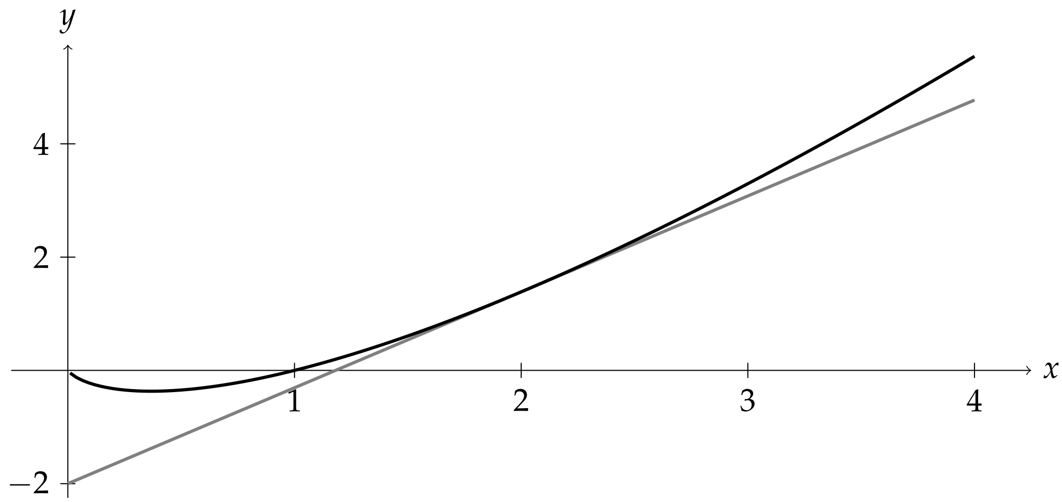
Solution: We find the y-coordinate on the graph at $t = 2$ to be $P(t) = 2\ln 2$. By the product rule, the derivative of $P(t)$ is

$$P'(t) = 1 \ln t + t\frac{1}{t} = \ln t + 1,$$

so $P'(2) = 1 + \ln 2$. Using the point-slope formula, the tangent line is given by

$$y = (1+\ln 2)(t-2) + 2\ln 2 = (1+\ln 2)t - 2.$$

We graph both $P(t)$ and $Q(t) = (1+\ln 2)t - 2$ below between $t = 0$ and $t = 4$:



3.R.66. Given that $r(2) = 4$, $s(2) = 1$, $s(4) = 2$, $r'(2) = -1$, $s'(2) = 3$, and $s'(4) = 3$, compute the following derivatives, or state what additional information you need to be able to compute the derivative.

(a) $H'(2)$ if $H(x) = r(x) + s(x)$

(b) $H'(2)$ if $H(x) = 5s(x)$

(c) $H'(2)$ if $H(x) = r(x) \cdot s(x)$

(d) $H'(2)$ if $H(x) = \sqrt{r(x)}$

Solution (a): Since $H'(x) = r'(x) + s'(x)$, $H'(2) = r'(2) + s'(2) = -1 + 3 = 2$. ■

Solution (b): Since $H'(x) = 5s'(x)$, $H'(2) = 5s'(2) = 5(3) = 15$. ■

Solution (c): Since $H(x) = r(x)s(x)$, $H'(x) = r'(x)s(x) + r(x)s'(x)$ by the product rule. Then

$$H'(2) = r'(2)s(2) + r(2)s'(2) = (-1)(1) + (4)(3) = -1 + 12 = 11.$$

Solution (d): Since $H(x) = \sqrt{r(x)} = (r(x))^{1/2}$, $H'(x) = \frac{1}{2}(r(x))^{-1/2}r'(x) = \frac{r'(x)}{2\sqrt{r(x)}}$, so

$$H'(2) = \frac{r'(2)}{2\sqrt{r(2)}} = \frac{-1}{2\sqrt{4}} = -\frac{1}{2(2)} = -\frac{1}{4}.$$

Extra Problem. Let $f(x) = \sqrt{x^2 - 1}$.

(a) Find the first derivative of $f(x)$. Simplify your answer.

(b) Find the second derivative of $f(x)$. Simplify your answer. (Hint: use the quotient rule and your answer to part (a).)

Solution (a): Since $f(x) = (x^2 - 1)^{1/2}$, $f'(x) = \frac{1}{2}(x^2 - 1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 - 1}}$. ■

Solution (b): Using the quotient rule, we take the derivative of $f'(x)$, reusing part (a):

$$f''(x) = \frac{\sqrt{x^2 - 1} - x\frac{x}{\sqrt{x^2-1}}}{(\sqrt{x^2 - 1})^2} = \frac{\sqrt{x^2 - 1} - \frac{x^2}{\sqrt{x^2-1}}}{(\sqrt{x^2 - 1})^2}$$

We can also simplify away the nested fraction in the numerator:

$$f''(x) = \frac{\sqrt{x^2 - 1} - \frac{x^2}{\sqrt{x^2-1}}}{(\sqrt{x^2 - 1})^2} \cdot \frac{\sqrt{x^2 - 1}}{\sqrt{x^2 - 1}} = \frac{x^2 - 1 - x^2}{(\sqrt{x^2 - 1})^3} = -\frac{1}{(\sqrt{x^2 - 1})^3}.$$