Decimation-in-frequency Fast Fourier Transforms for the Symmetric Group

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The Discrete Fourier Transform

The DFT reveals structure that is **invariant** under symmetry.

Suppose we want to analyze some periodic signal f:

- We pick an arbitrary full time period of f
- Take N samples $f_0, f_1, \ldots, f_{N-1}$ of f in this time period

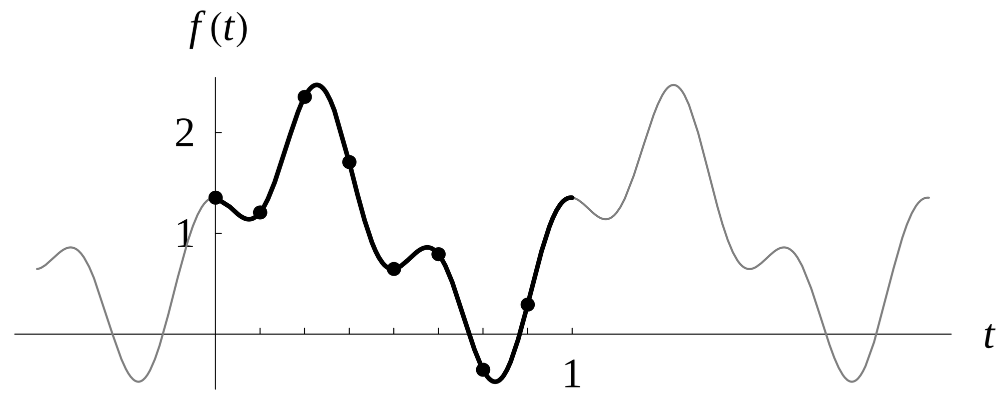


Figure 1: $N = 8$ samples of a periodic signal f.

For $0 \le k \le N-1$, define the **Discrete Fourier Transform** (DFT) to be

$$\hat{f}_k = \sum_{j=0}^{N-1} f_j \omega^{-jk} \quad \text{where } \omega = e^{2\pi i/N}.$$

The $\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_{N-1}$ are the **Fourier coefficients** of the samples $f_0, f_1, \ldots, f_{N-1}$.

$$\begin{pmatrix} f_0 \\ \vdots \\ f_{N-1} \end{pmatrix} \xrightarrow{\text{input}} \boxed{DFT_N} \xrightarrow{\text{output}} \begin{pmatrix} \hat{f}_0 \\ \vdots \\ \hat{f}_{N-1} \end{pmatrix}$$

Decomposition of original signal into "pure frequencies":

$$f(t) \approx \sum_{k=0}^{N-1} \underbrace{\hat{f}_k}_{\text{amplitude}} \left(\cos \frac{2\pi k}{N} t + i \sin \frac{2\pi k}{N} t \right)$$

("pure" frequency)

Our signal f above then decomposes as shown below:

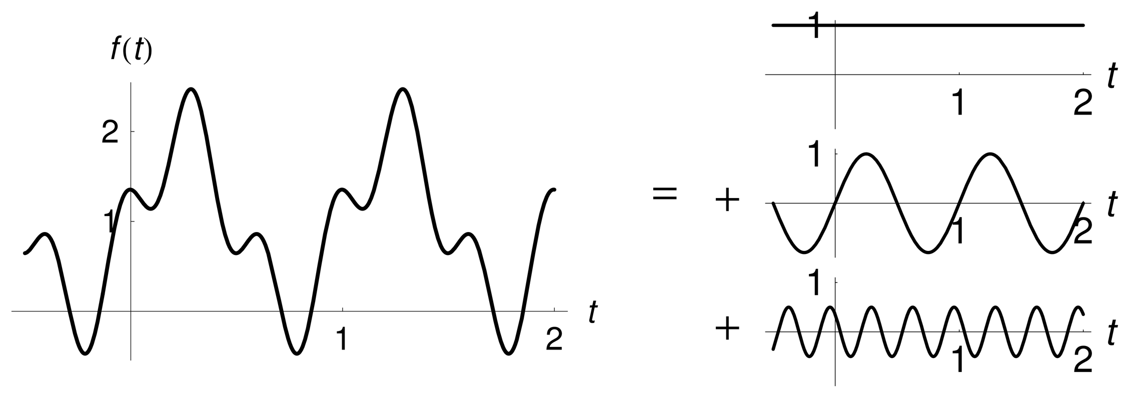


Figure 2: Fourier decomposition of f into three pure frequencies.

Suppose we sample our signal over a *different* time period

- The samples $f_0, \ldots, f_{N-1}$ could be much different
- But the Fourier coefficients $\hat{f}_k$ will not be
- The DFT is invariant under **translational** symmetry

Group-Theoretic DFTs

Different spaces have **different symmetries**:

Space	Symmetry	Group
time domain	time translations	$\mathbb{Z}/N\mathbb{Z}$
sphere	rotations about center	$SO(3)$
lists	permutations	S_n

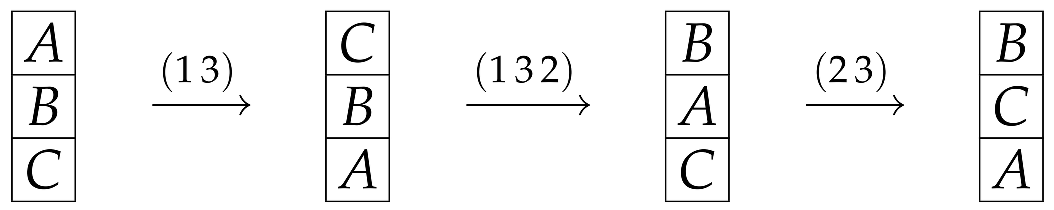


Table 1: Some different spaces and their associated symmetries

We therefore want **generalized DFTs** that show us similar **symmetry-invariant** structure. We can write these symmetries abstractly as **groups** and define these new DFTs using tools from **abstract algebra**:

$$\{\text{functions } f : G \to \mathbb{C}\} \longrightarrow \{f \in \text{group algebra } \mathbb{C}G\}$$

Wedderburn's Theorem *The group algebra $\mathbb{C}G$ of a finite group G is isomorphic to an algebra of block diagonal matrices:*

$$\mathbb{C}G \cong \bigoplus_{i=1}^{h} \mathbb{C}^{d_i \times d_i}.$$

For example, $\mathbb{C}S_3$ decomposes thus:

$$\mathbb{C}S_3 \cong \mathbb{C}^{1\times 1} \oplus \mathbb{C}^{2\times 2} \oplus \mathbb{C}^{1\times 1} = \begin{pmatrix} \bullet & & & \\ & \bullet & \bullet & \\ & \bullet & \bullet & \\ & & & \bullet \end{pmatrix}$$

Every $\mathbb{C}$-algebra-isomorphism $D : \mathbb{C}G \to \bigoplus_{i=1}^{h} \mathbb{C}^{d_i \times d_i}$ is called a **Discrete Fourier Transform** (DFT) for G. The coefficients of the matrix $D(f)$ are called the **(generalized) Fourier coefficients** of f.

The Problem

Naïve DFTs use N^2 operations

The Solution (for $\mathbb{Z}/N\mathbb{Z}$)

The Cooley-Tukey **Fast Fourier Transform** (FFT) computes classical DFT in $N \log N$ operations

Cooley-Tukey FFT uses factorization of the group $\mathbb{Z}/N\mathbb{Z}$:

$$1 < \mathbb{Z}/p_1\mathbb{Z} < \mathbb{Z}/p_1p_2\mathbb{Z} < \cdots < \mathbb{Z}/N\mathbb{Z}$$

Other groups G admit different subgroup chains:

$$1 = G_0 < G_1 < G_2 < \cdots < G_n = G$$

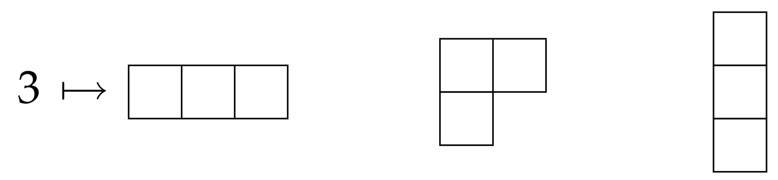
FFTs for the Symmetric Group

For the symmetric group S_n, we select the **subgroup chain**

$$1 = S_1 < S_2 < S_3 < \cdots < S_n$$

Blocks in matrix algebra for $\mathbb{C}S_n \leftrightarrow$ partitions of n:

$$\mathbb{C}S_3 \cong (\bullet) \oplus \begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix} \oplus (\bullet)$$



Paths through **character graph** index rows and columns of matrix algebra blocks:

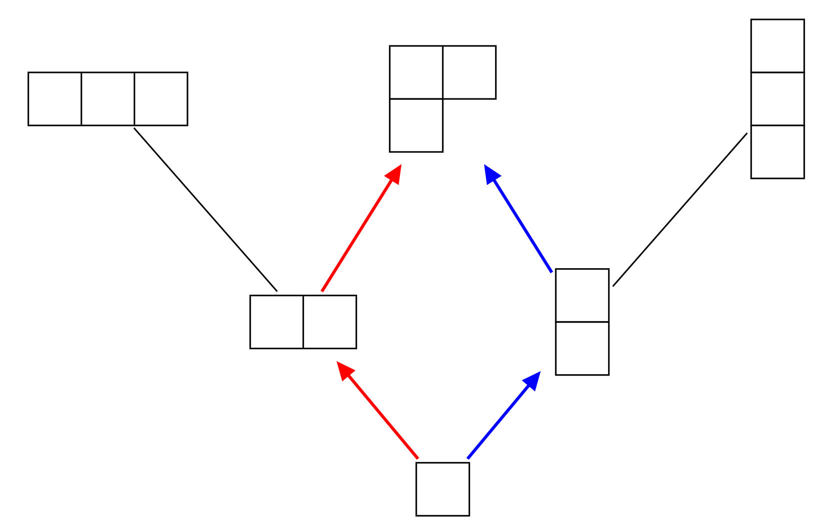


Figure 3: Character graph for $1 < S_2 < S_3$. Pair of paths shown indexes coefficient in first row and second column of second matrix block.

Decimation-in-frequency approach to FFT:

- Partial paths in character graph give subspaces of $\mathbb{C}S_n$
- At stage for S_k, we project onto these subspaces
- Build sparse factor from projections
- By stage for S_n, we have full paths
- Each pair of paths corresponds to a Fourier coefficient

Conjecture *The complexity of the evaluation of our decimation-in-frequency FFT for S_n is $O(n^2 n!)$. The complexity of the inverse transform is also $O(n^2 n!)$.*

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Results

We have computed **sparse matrix factorizations** of the DFT matrix for S_n for $n = 3$ to 6 using *Mathematica*.

n	$\oplus$	$\otimes$	t_n^{DIF}	t_n^{M}	$\frac{1}{2}n(n-1)$
3	14	4	2.7	2.7	3
4	112	42	5.3	5.4	6
5	966	424	8.8	9.1	10
6	9278	4631	13.8	13.6	15

Table 2: Operation counts for the evaluation of the decimation-in-frequency FFT. Here, t_n^{DFT} denotes the reduced complexity of our decimation-in-frequency algorithm for S_n, while t_n^{M} denotes the reduced complexity of Maslen's decimation-in-time algorithm.

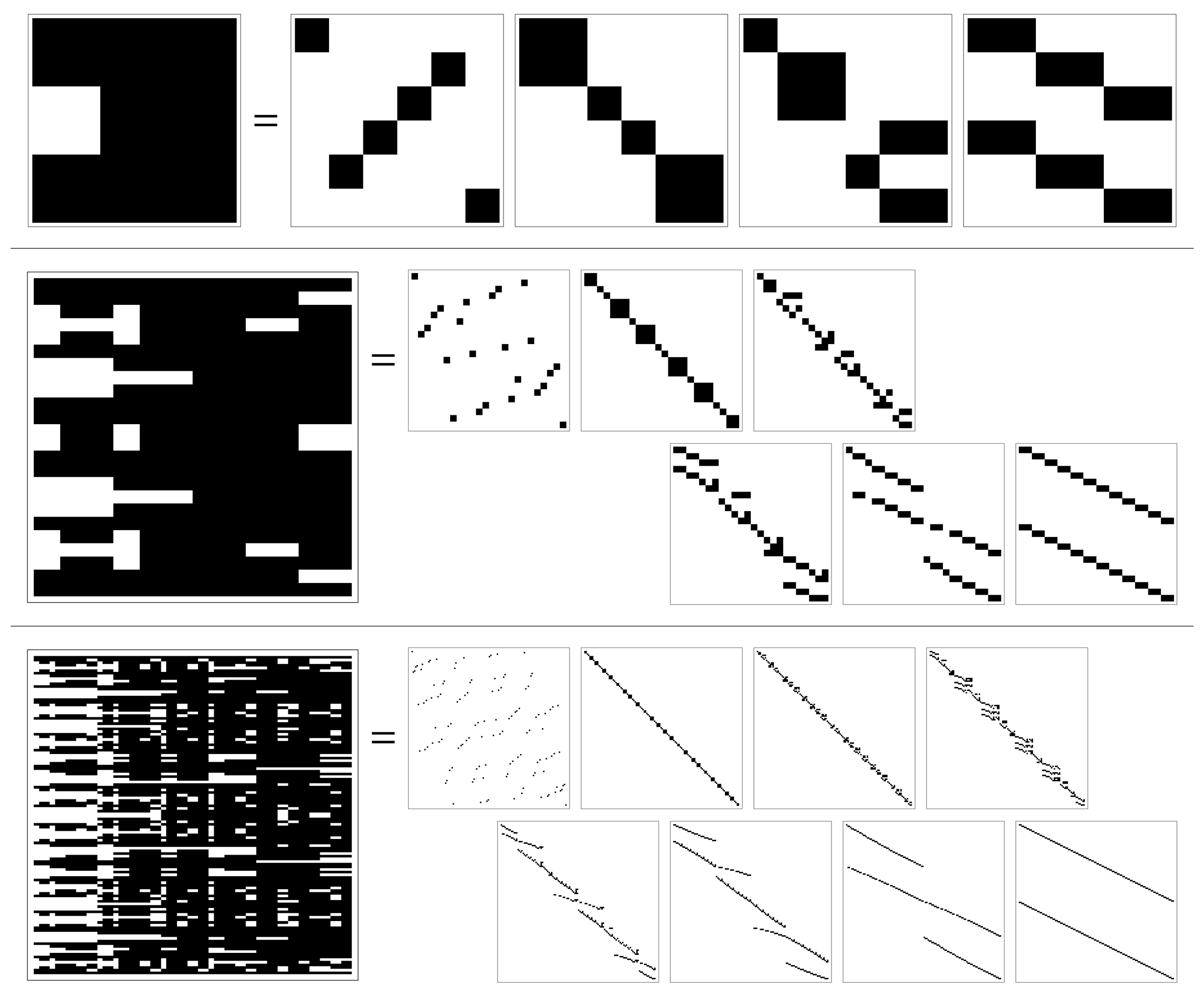

Figure 4: Sparse factorizations of DFT matrix for S_n for $n = 3$ to 5.

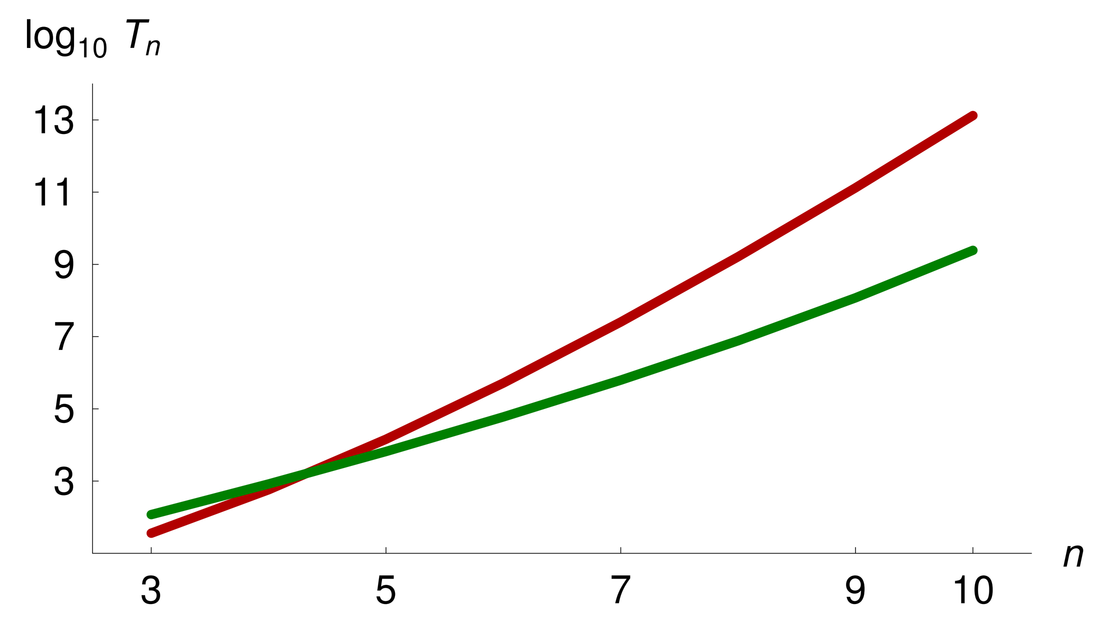

Figure 5: Comparison of costs for group algebra multiplication (red line) and FFT-based matrix algebra multiplication (green line). For $n \ge 5$, the FFT-based multiplication is more efficient.