M283
3/13/08
W#18

- Examples of CY- structures (conjectural)
  - Floer - theory - Gromov - Witten thy
  - string topology?

I. Floer thy, Gromov - Witten thy

(M, ω) a symplectic mfld, $\omega \in \Omega^2 M$, $d\omega = 0$,
nondegen: $\omega_x : T_xM \times T_xM \to \mathbb{R}$ nondegen.

Symplectic action: 'fn'l' $a : \widetilde{LM} \to \mathbb{R}$. - do Morse thy
'cover' of LM - ex: univ. cover

so let $\widetilde{LM}$ = univ. cover of LM - model: $\widetilde{LM} = \{(\gamma, \theta) : \gamma \in LM,\ \theta : D^2 \to M$ w/ $\partial\theta = \gamma\}/\sim$

where $(\gamma, \theta_1) \sim (\gamma, \theta_2)$ if $\theta_1 \#_\gamma \theta_2 : S^2 \to M$
is nullhomotopic. - over contractible loops - (component of LM).

so $a(\gamma, \theta) = \int_{D^2} \theta^* \omega$, (well-def by Stokes)

in fact, a defines on $\widetilde{LM}_\omega \to LM$ - $\mathbb{Z}$-cover of LM; where

$\widetilde{LM}_\omega = \{(\gamma, \theta)\}/\sim$, but $(\gamma, \theta_1) \sim (\gamma, \theta_2)$ if $\theta_1 \#_\gamma \theta_2$

has $\underline{\langle \omega, \theta_1 \#_\gamma \theta_2 \rangle = 0}$.
- assuming ω an integral form,
- takes values in $\mathbb{Z}$.

Now: crit pts of a correspond to constant loops, (and spheres).
- too numerous, not discrete, not nondegen.
- need metric - choose almost-C- structure ($J : TM \to TM$,
$J^2 = -id$) compat w/ ω; so
$\langle u, v \rangle = \omega(u, Jv)$ is pos-def. (hence giving
metric on M).

- contractible choice

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Gradient flow lines: $\gamma : \mathbb{R} \xrightarrow{\text{line}} LM$ satisfying

or $S^1 \times \mathbb{R} \to M$ converging to constant loops (since $M \subset pt?$)

$\searrow_{S^2} \nearrow$

- make $\gamma|_{S^2}$ J-holomorphic: $D\gamma : T\mathbb{P}^1 \to TM$ preserving J.

- hard to calculate ... not like traditional Morse thy - crit pt's not isolated.

perturb everything by a Hamiltonian: $H : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$
time dep, periodic.

- gives Ham. vector field : X_H defined by $\omega(X_H(t,x), v) = -dH_{(t,x)}(v)$.

(unique b/c of nondegeneracy of $\omega_{(t,x)}$)

New perturb action: $a_H : \tilde{LM} \to \mathbb{R}$: $(\gamma, \theta) \mapsto \int_{D^2} \theta^* \omega - \int_0^1 H(t, \gamma(t))\, dt$.

- now: crit pts are loops (+ extra data) $\gamma : S^1 \to M$

satisfying $\frac{d\gamma(s)}{dt} = X_H(s, \gamma(s))$. (periodic sol'ns of Ham eq'n)

choose: generic H w/ nondegen crit pts, flow lines satisfy perturbed C-R eq'ns:

$\phi : S^1 \times \mathbb{R} \to M$, w/ converging to crit loops,
(t, s)

$\partial_t \phi - J(\partial_s \phi - X_H(t, \phi(t,s))) = 0$ - still call J-holomorphic cyl's.

define Floer chain cx? $CF_q(M, \omega) \xrightarrow{\partial} CF_{q-1} \longrightarrow \cdots$,

where CF_q gen by crit pts of 'index q' ; bdy maps:

$\langle \partial[a], [b] \rangle = \# \mathcal{M}(a, b)$ (# of components of moduli connecting a, b.)

- i.e., # of J-hol cylinders connecting a to b).

get: $HF_*(M,\omega) = H_*M$, M closed mfld.

expect field thy $GW : \mathcal{O}e_\Lambda \to \mathrm{Comp}(\mathbb{Q})$ where

$GW(S^1) = CF_*(M,\omega)$.

Digression: benefits of Floer Theory:

- Poincaré-Hopf index thm: if X is any v.f., on cpt M, # of zeros of $X \geq \chi(M)$.

restrict: (Morse: if $X := \nabla f$, $f: M \to \mathbb{R}$ Morse fn, # of zeros of $\nabla f \geq \sum \mathrm{rk}\, H_i M$ $\geq \chi(M)$)

- $\mathcal{H}_0^f$ chain cx computing $H_i(X)$ w/ gens = crit pts.

analogous for diffeo-fixed pts:

Lefschetz: suppose $f: M \to M$ is a diffeo, $\partial M \neq \emptyset$, M cpt, any $f \simeq id_M$ (isotopic).

Then # of fixed pts of $f \geq \chi(M)$.

1990s.

Floer: proves analogue of this (Arnold's conjecture; proved most; Hofer, ... filled in rest)

(M,ω) symplectic, $f: M \to M$ a "symplectomorphism", &

f is "Hamiltonian isotopic" to id_M. Then # fixed pts of $f \geq \sum \mathrm{rk}\, H_i M$.

isotopic: recall Hamiltonian: $H: \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$, w v.f. X_H. $\exists$ fact Thm.:

$\phi: \mathbb{R} \to M$ w/ $\frac{d\phi}{dt} + X_H = 0$. Given $x \in M$, $\exists$ unique ϕ_x

satisfying DE w/ $\phi_x(0) = x$. - defined for all $t \in \mathbb{R}$ by M cpt.

Consider: $\phi^H: M \times \mathbb{R} \to M$, $\phi^H(x,t) = \phi_x(t)$.

- check each $\phi^H_t: M \to M$ a symplectomorphism; $\phi_0 = id$.

- isotopy from id to other symplectomorphism.

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Then $f: M \to M$ Ham. isotopic to id, if $\exists$ Ham H s.t. $f(x) = \phi^H(x, $ MAR 13 2008

Suppose x a fixed pt: the $\tilde{d}(t): \mathbb{R} \to M$ has $d_x(0) = d_x(1) \circ x$,

hence defines $\mathbb{R}/\mathbb{Z} \xrightarrow{\tilde{d}}$ M. map w/ $\frac{d\tilde{d}}{dt} + X_H = 0$.

i.e., d_x a crit pt of Floer fn'l. → generator of Floer chains.

since $HF_* M_1 \cong H_* M$, proving result.

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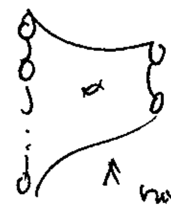
back to field theories:

Σ R.S:  $\xrightarrow{\text{J-holo}} M^{2n}$. J- ... chosen

has cylindrical ends:

– $a_1, \dots, a_p$, $b_1, \dots, b_q$ are crit pts of a_H.

moduli space: $\mathcal{M}(\Sigma; \vec{a}, \vec{b}) := \{ d : \Sigma \to M$ w/ (stable) J-holo that converges w/ finitely to a_i on 'incoming' bdy components, b_j on 'outgoing' $\}$.

extra sing's ok.

$\dim \mathcal{M}(\Sigma; \vec{a}, \vec{b}) = n\chi(\Sigma) + \left(\sum_i \text{ind } a_i - \sum_j \text{ind } b_j\right)$.

~~if dim = 0~~, ... define $CF_*^{\otimes p} \xrightarrow{\mu_\Sigma} CF_*^{\otimes q}$ by: $\mu_\Sigma(a_1, \dots, a_p) = \sum \#\mathcal{M}(\Sigma, \vec{a}, \vec{b}) \, |\vec{b}|$,

sum over all $\vec{b}$ w/ $\dim \mathcal{M}(\Sigma, \vec{a}, \vec{b}) = 0$.

→ defines topological field thy. ; pair of pants → $\mu : CF^{\otimes 2} \to CF^{\otimes 1}$
multiplication

GW theory: allow Σ's to vary over moduli space of R.S's

(or,: GW theory is a TCFT. – haven't described open part.

– same story: holo discs ; RS's w/ open bdys.

• Associative (exotic) CY A_∞-category: conjectured to be the Fukaya category of (M,ω). (Aw)

• 'Fuk(M)' D-branes: Λ = Lagrangian submanifolds of M.:

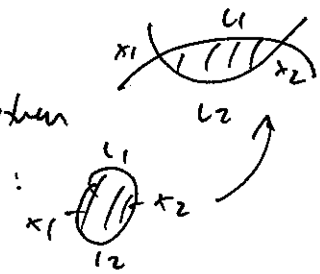
– $M = M^{2n}$, $L^n \subset M^{2n}$ s.t. $\omega|_{L^n} \equiv 0$. (isotropic submfld). Lagrangian

describe morphisms: $\mathrm{Hom}(L_1, L_2)_* = CF^*(L_1, L_2)$ – Floer cx.

w/ $CF^*(L_1, L_2)$ gen by intersection pts $L_1 \cap L_2$ – isolated pts? – what if not transverse?

$\langle \partial [x_1], [x_2] \rangle = \# \mathcal{M}(\{x_1\}, \{x_2\})$ J-holo.

$\mathcal{M}(x_1, x_2)$ = moduli space of $\{ \phi : D^2 \to M$: between $x_1, x_2 \}$.:

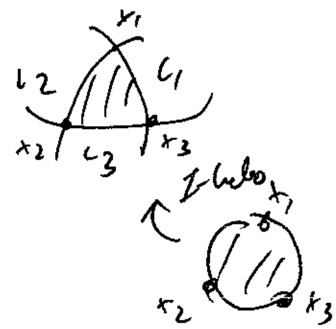


composition: $\mathrm{Hom}(L_1, L_2) \otimes \mathrm{Hom}(L_2, L_3) \xrightarrow{m_2} \mathrm{Hom}(L_1, L_3)$

$CF^*(L_1, L_2) \otimes CF^*(L_2, L_3) \to CF^*(L_1, L_3)$:

$x_1 \otimes x_2 \mapsto$

$\langle m_2(x_1, x_2), x_3 \rangle = \#$ holo triangles:



issues: $*$-invariance; analytic issues about gluing.

Conj: • $HH_*(\mathrm{Fuk}(M)) \overset{?}{=} HF_* M = H_* M$. ($c_1(M)=0$ – CY cond. Kaehler)

– worked out in a few (concrete) cases. (– P. Seidel). → see Abouzaid's minicourse.

• recall CY – descr. in terms of cat of coh. sheaves on a CY variety: conj that Fuk(M), cat of sheaves are Morita equivalent (mirror symmetry conjecture).

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String topology: purely topological analogue of this: — object noted M_d closed, oriented

- exists a 'homological' TQFT' : $S : OC_\Lambda^{hom} \to$ Graded $\mathbb{Z}$-mod: ($\mathbb{k}$?)

- homological: morphisms are Gr-$\mathbb{Z}$-mod — pass to homology, moduli spaces.

$$OC_\Lambda^{hom} = H_*(M(a,b)).$$

- stipulate $S(\mathbb{I}) = H_*(LM)$; (really a thm — lots of work done in closed setting, gen. too). — pf by Hopkins-Lurie ? ; Godin?) — see her paper?)

- let Λ = all closed submflds of M, any dim'n.
- try to do on chain level (Costello)

$$\mathrm{Hom}(L_1, L_2) = C_*(P(L_1, M, L_2))$$

$$({}_{L_1}P_{L_2})^{tr} = \{\gamma : [0,1] \to M : \gamma(0) \in L_1, \gamma(1) \in L_2\}$$

Mult. (on H_*) is

$$\begin{tikzcd}
{}_{L_1}P_{L_2} \times {}_{L_2}P_{L_3} \arrow[d] & {}_{L_1}P_{L_3} \arrow[l, "\text{codim} = \dim L_2"'] \arrow[r] \arrow[d, "\text{p.s.?}"', "\text{any } 1/2 \text{ in } L_2"] & {}_{L_1}P^{L_2}_{L_3} \\
L_2 \times L_2 & L_2 \arrow[l, "\Delta"] &
\end{tikzcd}$$

Gives ${}_{L_1}P_{L_2} \times {}_{L_2}P_{L_3} \xrightarrow{\text{Thom coll.}} (P_{L_1,L_2,L_3})^{TL_2}$ — descends to homology:

$$H_*\, {}_{L_1}P_{L_2} \otimes H_*\, {}_{L_2}P_{L_3} \xrightarrow{H_*(\text{coll})} H_{*+\ell_2} \xrightarrow{\mathrm{Thom}(i)} H_{*+\ell_2}\, {}_{L_1}P_{L_3}$$

- still an issue on chain level.

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Conj: • These data define an A_∞-cat $\mathcal{S}_{?}(M)$,

(using chains somehow); + $HH_*(\mathcal{S}(M)) \cong H_* LM$.

idea: chain level gymnastics. (me?)

${}_{L_1}P_{x_0} \times_{x_0} {}_{x_0}P_{L_2}$ $\quad {}_{L_1}P_{L_2} \xrightarrow{\text{? ?}} M$, fib'n. - fiber over x_0 ?:

$\cong$

& ${}_{L_i}P_{x_0} \simeq \text{hofib}(L_i \to M)$ - F_{L_1}, ~~F_{L_2}~~

$F_{L_1} \times F_{L_2} \to {}_{L_1}P_{L_2} \to M$ - Loop: $\Omega M \to F_{L_1} \times F_{L_2} \to {}_{L_1}P_{L_2}$.

So: up to h.t.py, a princ. ΩM- bundle E (G_M)

Then ${}_{L_1}P_{L_2} \simeq EG_X \times_{G_X} (F_{L_1} \times F_{L_2})$

so $C_*({}_{L_1}P_{L_2}) \simeq C_*(F_{L_1}) \otimes^L_{C_*(G_X)} C_*(F_{L_2})$

Conj: PD gives an ~~[illegible]~~ QI: $C_*({}_{L_1}P_{L_2}) \longrightarrow {}^R\text{Hom}^{(k-d)}_{C_*\Omega X}(C_*F_1, C_*F_2)$

- we can compose (M s-can things)

get back to ~~[illegible]~~ $H_*(M)$? via Morita equivalence.

$HH^*(C^*M)$ vs. $HH^*(C_*\Omega X)$ - 2 examples of path spaces;

$\Omega M = + P_{*}*$; $C^*M \simeq C_*\text{End}^{C_*}(MPM)$ - expect for

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