M 283
2/21/08
LN #13

Today: Costello paper: homological algebra for categories:

- recall statement of thm: $\Lambda$: set of D-branes - labels for 1-manifolds;
- define OC cob. category $\mathcal{OC}_\Lambda$, sub cats $\mathcal{O}_\Lambda \overset{i}{\hookrightarrow} \mathcal{OC}_\Lambda \hookleftarrow \mathcal{C}$.
- study $\mathrm{Fun}^{\otimes}_{\text{h/c, -linear}}(\mathcal{OC}_\Lambda, \mathrm{Ch}(k))$ (TCFTs), ($k$ char $k = 0$) H-L: generalize to any SMC in range.

(Moore-Segal) TQFT w/ $\Lambda = \{*\}$ given by FA $A = F(I)$, + comm. FA $B$ ~~assoc to~~ $F(S^1)$, + maps $i_*: B \to Z(A)$, & vice versa, $i^*: B \to B$ $c_i$

+ Cardy condition $i_* i^* v = \sum_i q^i v q_i$ - shrinking.

Costello: views $i^*, i_*$ as restriction ~~fctrs.~~ / induction ~~fctrs.~~ functors.

$$i^*: \mathrm{Fun}^{\otimes}(\mathcal{OC}_\Lambda, \mathrm{Comp}/k) \longrightarrow \mathrm{Fun}^{\otimes}(\mathcal{O}_\Lambda, \mathrm{Comp}/k)$$

vs. $i_*: \mathrm{Fun}^{\otimes}(\mathcal{O}_\Lambda, \mathrm{Comp}/k) \longrightarrow \mathrm{Fun}^{\otimes}(\mathcal{OC}_\Lambda, \mathrm{Comp}/k)$ ← left adjoint.

consider:
- algs $A_1 \overset{i}{\hookrightarrow} A_2$; have res map $i^*: A_2\text{-mod} \to A_1\text{-mod}$;
  and induction map $i_*: A_1\text{-mod} \to A_2\text{-mod}$.
  adjoint $i_* M = A_2 \otimes_{A_1} M$.

problem: $i_*$ not exact (is right exact).

Thm: (Costello) 1) The cat. of TCFTs ($\mathrm{Fun}^{\otimes}(\mathcal{OC}_\Lambda, \mathrm{Comp}/k)$) is htpy equiv to extended $A_\infty$-calabi-yau categories. (data on open part)

2) ~~for any TCFT $\phi$~~, there is a left adjoint to res $i^*$:

$$(\mathbb{L}i)_*: \mathrm{Fun}^{\otimes}(\mathcal{O}_\Lambda, \mathrm{Comp}/k) \longrightarrow \mathrm{TCFTs}$$

~~pt.~~ 3) ~~Does~~ $((\mathbb{L}i)_*(\phi))(S^1)$ has homology $HH_*$(CY cat) (underlying) of $\phi$.

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define some terms:

1) Calabi-Yau categories: (gen of FA) – cat enriched over Complx;

category $\mathcal{C}$, $\Lambda$ := obj $\mathcal{C}$. for each $A \in \Lambda$, have a trace map

$Tr_A : Hom(A,A) \to k$ (w/ duality?),

$Tr_{A,B} : Hom(A,B) \otimes Hom(B,A) \to k$ symmetric; nondegen.

(essentially FA in case w/ $\Lambda = \{*\}$)

satisfy the following similarly symplectic note. in

CY-relation: setting of GW theory; cat. is $C_*(M)$

if M, $M$ is calabi-yau, satisfy nondegen condition.

Assoc: assoc. up to htpy.

HH statement: is $B \to Z(A)$ statement in TQFT case.

– center exists in TQFT theory;

– recall: center of alg A = $HH^0(A,A)$ (complete cat.)

$A \xrightarrow{\delta} Hom(A,M)$:
$\delta(a)(x) = ax - xa$ – so a a cycle iff $a \in Z(A)$.
– no sols

• work w/ DG sym. monoidal cats: (dgsmc's)

– enriched over Complx – morphism spaces are chain cxs.;
also symm. monoidal cat. – where $\otimes$ is $\amalg$ cat.; maps (assoc).

$Hom(\alpha, \alpha') \otimes Hom(\beta, \beta') \longrightarrow Hom(\alpha \amalg \beta, \alpha' \amalg \beta')$

Ex: Comph; tqft.

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Given a DGSM $A$, a left $A$-module is a monoidal functor $F : A \to \mathrm{Cplx}_k$.
(right: $A^{op} \to \mathrm{Cplx}(k)$).

$A$ is a bimodule: ($A \otimes A^{op}$-module).

$A : A \otimes A^{op} \to \mathrm{Cplx}/k \quad a \otimes a' \mapsto A(a', a)$. (s monoidal).

- exact functors: ~~[illegible]~~, if cat $\mathcal{C}$ has a notion of quasi-isom (like $\mathrm{Ch}(k)$) : (collection of morphisms containing isos, closed under composition), generates an eq. relation on objects:

$A, B$ are quasi-isom if $\exists$ zig-zag of qi's connecting them.

- a functor $F : \mathcal{C} \to \mathcal{D}$ is exact if it takes QIs to QIs.
- (can talk about quasi-isomorphic functors: NT ~~[illegible]~~ consisting of QIs.)

- defn: a quasi-equivalence : $\mathcal{C} \to \mathcal{D}$ is a ~~[illegible]~~ functor $F : \mathcal{C} \to \mathcal{D}$ w/ a functor $G : \mathcal{D} \to \mathcal{C}$ s.t. $FG$, $GF$ are QI to id's.

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Tensor products: Let $A, B$ be dgsm's; suppose $M$ a $B$-$A$-bimod. (left $B$-mod, right $A$-mod, compatibly). Let $N$ be a (left) $A$-mod.

want to define (L) $A$-mod $M \otimes_A N$ :

- define $(M \otimes_A N)(b)$ : (chain cx) $b \in A$, s.t. for each $a$, has map (union)

~~$A(b, a) \otimes (M \otimes_A N)(b) \to M \otimes_A N$~~

$M(b, a) \otimes_k N(a) \xrightarrow{f} (M \otimes_A N)(b)$, universal wrt that

$$\begin{array}{ccc} M(b,a) \otimes_k A(a,a') \otimes_k N(a') & \xrightarrow{a \otimes \mathrm{id}} & M(b,a') \otimes N(a') \\ \downarrow {\scriptstyle \mathrm{id} \otimes a} & & \downarrow {\scriptstyle f} \\ M(b,a) \otimes N(a) & \xrightarrow{f} & (M \otimes_A N)(b). \end{array}$$

commutes (coend constr).

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• consider $f: A \to B$ of alg[ebra]s. have $f_*: A\text{-mod} \to B\text{-mod}$:

$M \mapsto B \otimes_A M$. ($B$ a $B$-$A$-bimod by $f$). & $f^*: B\text{-mod} \to A\text{-mod}$ forget

, easy to see $f_*$ L Adj to $f^*$; but $f^*$ is exact; $f_*$ need not be.

– resolve this. (derived fts/algebra).

**Defn**: An $A$-mod $M$ is flat if $- \otimes_A M$ is exact. ( Mod-$A \to$ ~~cat~~ $Ch(k)$) cofibrant conditions $B$-mod-$A \to (B\text{-mod})$, etc

Let $A$-flat = cat of flat $A$-mods. (subcat.)

**Thm**: Suppose $A$ is a DGSM, w/ strict monoidal struct; & has a unit obj $u \in A$:

& the monoid ob $A$ is free (details later – essentially free cat on monoids),

then $\exists$ functor $F$: $A$-mod $\to$ $A$-flat, s.t. $F \circ i$, $i \circ F$ are

quasi-isom to id. ($i$: $A$-flat $\to$ $A$-mod).

– replace module w/ flat one. (ex: replace $X^{\otimes n}$ w/ $\tilde{X}^{\otimes n}$ w/ free action).

Idea: preserve homology type w/ $\text{flat}^M$($M \otimes_A -$ exact).

**Defn**: Let $\text{ob} A \hookrightarrow A$ be a sub DGSM: need to keep

permutation morphisms / give $\Sigma_n$-actions.

– actually, do have free $\Sigma_n$-action on $a_1 \otimes \cdots \otimes a_n$

– groupoid: all morphisms are isomorphisms.

**Properties**: Lemma: view tensor prod $- \otimes_{\text{ob}A} -$ as a functor: $(\text{ob}A\text{-mod}) \times (\text{mod-ob}A) \to Ch(k)$.
$(B, C) \mapsto C \otimes_{\text{ob}A} B$)

is exact in each variable.

(need char $k = 0$; $\text{ob}A$ a groupoid; symmetries of an object are $\Sigma_n$.

– coinvariants of ~~$k$~~ $k\Sigma_n$ is exact, when char $k = 0$.

naturally: coinvariants does preserve HEs. – ~~$H$~~ $H^i(\mathbb{Q}\Sigma_n, -) = 0$, $i > 0$.

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Lemma: (construction) Let $C = \mathrm{Fun}(\mathrm{Ob}\,A, \mathrm{Ch}(k))$.

have $\mathrm{Ob}A\text{-mod} \xhookrightarrow{G} C$. (forgetful). Then $G$ has a left adjoint

$F : C \to \mathrm{Ob}A\text{-mod}$, which is exact.

( direct construction: - S/c $\mathrm{Ob}A$ a free monoid )

- specify on generators: - $a = a_1 \otimes \cdots \otimes a_k$ uniquely (up to iso) gens.

- given set $I \subset \{1, \ldots, n\} = [n]$; $a_I = \bigotimes_{i \in I} a_i$. (was on gens)

- given $V \in C$, define.

$$F(V)(a) = \bigoplus_{k \geq 0} \Big( \bigoplus_{\lambda \vdash [n]} V(a_{\lambda_1}) \otimes \cdots V(a_{\lambda_k}) \Big)_{S_k}$$

partitions w/ length $k$ — length. unordered partition.

- check monoidal. ; $I$ empty: $a_\emptyset$ = unit. in the cat.

No class Tues, 2/26.

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