M283
2/14/08
LN#11

Today: 1^{st} - correct formula for
- easier calculation

(*) choose $\bar{?}_n$ to RB_k)

- recall $F(\mathbb{R}^2, k) \simeq K(PB_k, 1) = B\,\mathrm{Diff}^+(D^2_k, \partial)$ (upto k pts)

- consequences for $\mathrm{Diff}^+(\odot, \partial + \text{pts})$ components, $\simeq k$.

Arnold result; have maps $F(\mathbb{R}^2, k) \to \mathbb{R}^2 \setminus 0 \simeq S^1$

$z_1, \dots z_k \mapsto \dfrac{z_i - z_j}{\|z_i - z_j\|}$

induces cohom class $\omega_{ij} \in H^1(F(\mathbb{R}^2, k))$.

Thm: $H^k(F(\mathbb{R}^2, k); \mathbb{Z})$ gen by $\omega_{ij} \in H^1(-)$,

- relations are • $\omega_{ij} = \omega_{ji}$

• $\omega_{ij}\,\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\,\omega_{ij} = 0$

- really not so bad: easy to calculate cohom (SS?)

Prop (f. cohom) - consider fibrations $\overset{\mathbb{R}^2 \setminus (k-1 \text{ pts})}{\simeq \bigvee^{k-1} S^1} \hookrightarrow F(\mathbb{R}^2, k) \overset{p_k}{\twoheadrightarrow} F(\mathbb{R}^2, k-1)$

(works for $\mathbb{R}^n$ as well)

- fibration has sections: map σ w/ $p_k \sigma \simeq id$ (canonically):

(homotopy)

- identify $\mathbb{R}^2 \xrightarrow{\varphi} \mathbb{R}^2_+$ (upper ½-plane); induces dotted

$F(\mathbb{R}^2, k-1) \underset{\simeq}{\xrightarrow{\varphi_*}} F(\mathbb{R}^2_+, k-1) \to F(\mathbb{R}^2, k)$

$(x_1, \dots, x_{k-1}) \mapsto (x_1, \dots, x_{k-1}, (0, -1))$ x_k

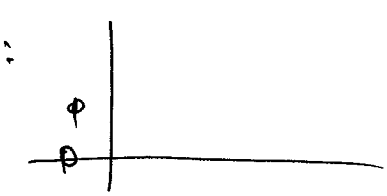
(≃) - gives section

so homotopy is surjective;

Serre spectral sequence for $H^*_* E$ then collapses; (b/c

(of fiber structure & - dim 0 or 1):

- level-0 stuff has to survive by surjectivity



→ cohom $H^k(F(\mathbb{R}^2, k)) \simeq \bigotimes_{j=1}^{k-1} H^k(\bigvee^{j-1} S^1)$ - ; identify as:

→ ⊗'s of algs; local coeff. issues? (wrong)

74 TFT

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• not obvious there are no π_1 - issues : $F(\mathbb{R}^2, n)$

– $\mathcal{B}$ obvious? $\pi_1(\mathcal{B}) = PB_{n-1}$; $\underline{\text{curr.}}$ $\underline{\text{action}}$ on ~~[illegible]~~ via $S_k \mathcal{B}$

$\pi_1 \mathcal{B}$ action on $\underline{H_*(F)}$ should be trivial; trivial (see F. cohen paper).

be trivial $\cong A\underline{\pi_1^{ab}(F)}$.

New: Work of K. Costello; Cheng, Moerdijk(?) & Segal : classification of

$\underline{\text{open-closed}}$ TCFTs & TQFTs. ($\underline{\text{MrS}}$: o-c 2D-TQFTs)

Costello: TQFTs $\Rightarrow HH^*(A)$ appears.

$\underline{\text{Results}}$ 1st : recall defn of a TCFT : let M = Segal group.

of Riemann surfaces;

• obj M = finite ordered sets (rel); (corr to ordered copies of S^1)

• morph $M(I, J)$ are moduli spaces of Riemann surfaces, that are cobordisms from $\coprod_I S^1$ to $\coprod_J S^1$.

$\underline{\text{Apply}}$ $C_*(-) : C_*(M)(I, J) = C_*(\mathrm{Map}(I, J))$.

recall a TCFT is a (lax?) monoidal functor $C_*(M) \xrightarrow{F} \mathrm{End}(V_*)$

some V_* is chain cx, obj's secn? arrangement.

want: $\underline{\text{quasi-isoms}}$ (H_*) $F(I \sqcup J) \xrightarrow{\simeq} F(I) \otimes F(J)$.

– too strict to $\underline{\text{require}}$ ~~[illegible]~~ chain $\underline{\text{Bun}}$. – need S_k of EZ/AW $\underline{\text{equiv}}$.

$\underline{\text{if}}$ F is $\underline{\text{B}}$ on Bun; Costello calls this a $\underline{\text{split}}$ TCFT then;

Lurie calls this $\underline{\text{strict}}$.

• could also conceive of $\underline{\text{higher-category}}$ structures (n-Lurie).

Produce $\underline{\text{open-closed}}$ version of these TCFTs :

let N_1, N_2 be n-mflds w/ bdy; (perh. w/ emb.) a cobordism W

btwn $N_1 \amalg N_2$ is an $\underline{\text{n-mfld}}$ w/ a decomposition of its $\underline{\text{bdy}}$

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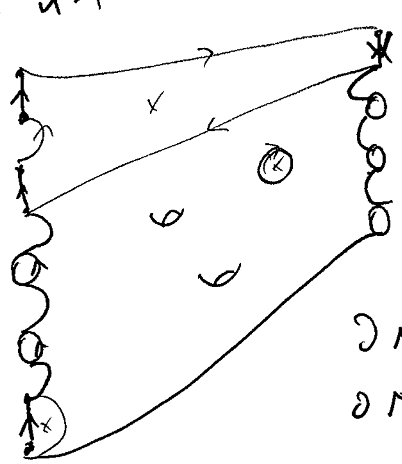
$\partial W = \partial_{in} W \cup \partial_{out} W \cup \partial_{free} W$, where $\partial_{free} W$ is a cobordism from $\partial(N_1)$ to ∂N_2.

$\partial_{in} W = N_1$, $\partial_{out} W = N_2$, $\partial_{free} W$: physicists' term

Ex: focus on $n=1$ case: oriented 1-mflds w/ 0-mfld bdy;

Ex:

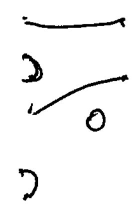


physicists = 'free' bdy:

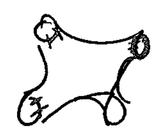
topological type: 8 bdy circles, + genus 2

∂N_1 = 6 pts;

∂N_2 = 2 pts; $\partial_{free} W$ = 1-mflds:



- not smooth? just drawn this way; can of course draw as mfld w/ corners on bdys of smooth Σ w/ bdy.



etc.

origin of free mflds? fns on end bdy mflds, apply alg top constructions to mflds; bdy on mflds ⇝ bdy conditions on structures

Ex: A field theory is a $\otimes$-functor from a SMC $C_*(M^{oc})$ to $Ch(k)$, where objects of M^{oc} are 1-mflds w/ bdy;
- know S^1-assignment; I-assignment? path spaces w/ bdy conditions

String top: $E(S^1) = H_*(LM)$; so $E(\amalg_n S^1) = H_*(Map(\amalg_n S^1, M)) \cong (H_*(LM))^{\otimes n}$;

& $E(I) = H_*(Map(I, M))$; but this is $\simeq M$ unless specify bdy conds: conditions: called "D-branes".

- Let $L_1, \dots, L_n \subset M$ be submflds; label bdy pts by L_i's.

Then $E(\overset{2}{\bullet}\!\!-\!\!\!-\!\!\overset{3}{\bullet}) = H_*(\{\gamma \in Map([0,1], M) : \gamma(0) \in L_2, \gamma(1) \in L_3\})$.

- relevance to knots^K, links^L in S^3 (.4 a 3-mfld; cf. to SFT.

- so $\mathcal{E}(\overset{L}{\bullet}\!-\!\overset{L}{\bullet})$ gives paths starting & ending on $L \cup a$.

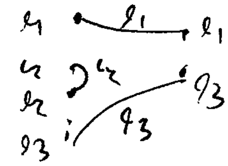
hence - introduce labeling data: let Λ = set of labels;

(lab. corresponds to category); define M_Λ to have objects:

pairs $(\mathcal{O}, \mathcal{C})$ of finite sets; ($\mathcal{O}$ = open pieces (I's); $\mathcal{C}$ = closed pieces (S^1's)

of labels s.t.: $\mathcal{O} \to \Lambda$. (labels, branes?)

morphisms: Riemann surfaces Σ w/ O/C bdys, (i.e., choice of decomps of $\partial\Sigma$)



s.t. free bdy components also labeled by Λ, compatibly w/ labels s.t. $\mathcal{O} \to \Lambda$.

composition is gluing, respecting labels. - free bdy curves = windows.

An open-closed TCFT is a $\otimes$-functor $C_*(M^{oc}) \to Ch(k)$. ↑ Ab?

- 2-cat setup instead? Costello shows we can replace C_*M_Λ w/ chain htpic chains; (sing. chains vs. cell-chains?) - some flexibility here.

what about OC-TQFTs? - objects are obj M_Λ; take morphisms to be discrete sets corr. to diffeo classes of open-closed cobordisms (i.e., $\pi_0(M_\Lambda(\Sigma, J))$.. (diffeos preserve bdys ptwise; fix labels too).

note: windows can rotate; not ~~...~~ bdy circles touching in/out.

- for every pair $(l_1, l_2) \in \Lambda$, an OC TQFT F defines $F(I_{l_1}^{l_2}) = V_{l_1, l_2}$

& $F(S^1)$ = ~~...~~ A. ~~...~~ v.space;

& each diffeo class of O/C cob $\rightsquigarrow$ linear map $V \otimes A$'s $\to$ V's.

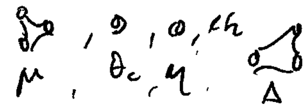
Ex: get map $V_3^1 \otimes A^{\otimes 2} \otimes V_4^4 \longrightarrow V_3^3 \otimes A^{\otimes 3}$

- incorporate closed field theory data → A still a comm FA;

Moore-Segal classification (when ∃ only 1 D-brane) of vector-space valued oc 2D TQFTs. ($\Lambda = 3\&3$) – can drop label.; ε a TQFT

2 pieces of data: $V \Leftrightarrow E(I)$; & $\varepsilon = E(S^1)$;

- ε a comm FA; ~~&~~ b/c of closed cobordisms + diffeos.



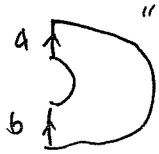
full sub cat satisfy 1 interval

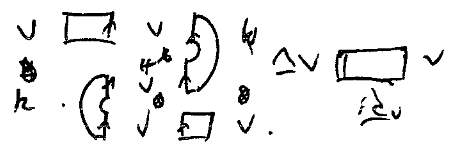
- open pieces: no circles (as in/out pieces):

- V is a (not nec. comm) FA:

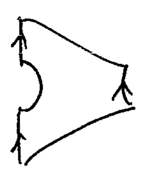
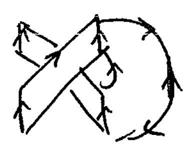
"drapes" – assoc; η_{open} θ_{open}.

 μ_0 unit &l  ; counit 

- FA? ~~non product~~ pairing $\langle , \rangle_0$:  "horseshoe" – duality? b/c of



not nec. comm:

 vs  – not diffeo as oriented mflds:

do see that $\langle a, b\rangle_0 = \langle b, a\rangle_0$ – symmetric:

 vs  : b/c can flip ~~...~~ RH bdy piece.

Thm: (Costello) non-comm FA A corresponding to this mult; HH^*A is a gr. comm FA; extends to a TCFT; universal in some sense for field theories w/ this

&. HH^*A an alg over $H_* B\,\mathrm{Diff}(P_n) \Leftrightarrow$ BV-alg (A a FEB FA) 2008

$\exists$ (alg) homs $i_K : \mathcal{E} \to Z(V) \hookrightarrow V$.

$i^K : V \to \mathcal{E}$ s.t.:

∂_K: from  "read morphisms"

S^1 I I (mix)

i^K:  — exercise: verify why $i_K(\mathcal{E}) \subset Z(V)$.

— homeomorphism takes same work:

① i_K a hom of algebras;

② i^K is adjoint to ∂_K under $\mathcal{E} \cong \mathcal{E}^*$, $V \cong V^*$ dualities; i.e.,

$$\langle i^* \phi, \alpha \rangle_c = \langle \phi, i_K \alpha \rangle_0 .$$

③ $\partial_K i^K : V \to Z(V) \subset V$ has formula:

$$i_K i^K(\phi) = \sum_{i=1}^{n} \psi^i \phi \psi_i$$ $\psi_1, \dots, \psi_n$ a basis for V; & ψ^i are dual basis w.r.t. $\langle -,- \rangle_0$.

(Cardy condition).

— hard to prove algebraically — draw right picture.

Moore–Segal: An O-C 2D TQFT w/ 1 D-brane is $\cong$ to

1) $\mathcal{E}$ comm FA, V an FA,

2) $i_K : \mathcal{E} \to Z(V)$,

3) i^K = adj of i_K, then have Cardy condition.

Pf: by Morse thy. decompose O-C-cobordisms appropriately. &

— piece together the P's, caps, discs, o/c discs.

+ flutes/reads $I \Leftrightarrow S^1$.