M283
2/5/08
W#8

Today: more coalgebra; back to field theory.

- Sketched Thm (Jones): if X is 1-conn, then $HH_*(C^*X; C^*X) \cong H^*LX$.

- when X a mfld, C^*X close to being a Frob. algebra via PD.
  - PD does not hold on the nose - just up to htpy.

- ingredients: simplicial desc. of S^1 $\rightarrow$ cosimplicial desc. of LX.

- write $S^1_k = \underline{k} := \{0, \dots, k\}$.

given a simplicial set $X_\bullet$, one has a chain cx C_* for computing $H_*(|X_\bullet|)$:

where $C_q = \mathbb{Z} \otimes X_q$. (free ab. gp gen by X_q);

& $\delta : C_q \rightarrow C_{q-1}$ is the alt. sum $\sum_{i=0}^{q} (-1)^i d_i$ of face maps.

from 282B: given a simplicial set, $\mathbb{Z} \otimes X_*$ gives a simplicial ab. gp;

- forming this chain cx computes $\pi_*(-)$ of this simplicial gp.
  - really the same as Dold-Thom Thm.

given a simplicial space (each X_q is a space & d_i, s_i maps are cts),

instead obtain a double cx for computing $H_*(|X_*|)$:

let $S_p(X_q)$ be the sing. p-simplices of X_q; (or cellular chains of X_q all cw)

have

$$\begin{array}{ccc} S_p(X_q) & \xrightarrow{d_{simpl.}} & S_p(X_{q-1}) \\ \downarrow d_{sing} & \partial & \downarrow \\ S_{p-1}(X_q) & \longrightarrow & S_{p-1}(X_{q-1}) \end{array}$$

- C_* of chain cxs; form double cx; D_*

Tot(D_*) computes $H_*(|X_\bullet|)$ as well.

Suppose G is a group, (perh. w/ topology); note G acts on $G^c = G$ on the right by conjugation: $g \cdot h = g^{-1} h g$ - study htpy orbits

$G^c \times_G EG$ ~~[illegible]~~ $= Ad(EG)$ - G-bundle over BG
- fiber bundle, not principal.

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intuit, for any prin. G-bundle $\begin{array}{c} G \to P \\ \downarrow \\ X \end{array}$, can form $\underline{Ad(P)} = G^c \times_G P$ . .

Claim: (q.v.) $Ad(P)$ is isomorphic to $Aut(P)$, where $Aut(P) \to X$ has fiber $Aut_G(P_x, P_x)$ over $x \in X$ – $P_x \cong G$, but not canonically.

so $Aut(-,-) \cong G$, but not canonically so.

Here, $\Gamma(Aut(P))$ = gp of automorphisms of P (bundle) over id_X

– often called the gauge gp of P.

$$\begin{tikzcd} P \arrow[r, "\phi"] \arrow[d] & P \arrow[d] \\ X \arrow[r, "id_X"] & X \end{tikzcd}$$

Thm: $\exists$ fiber htpy equiv $Ad(EG) \xrightarrow[\simeq]{d} LBG$:

$$\begin{tikzcd} G \arrow[r, "\cong"] \arrow[d] & \Omega BG \arrow[d] \\ G \times_G EG \arrow[r, "d"] \arrow[d] & LBG \arrow[d] \\ BG \arrow[r, "id"] & BG \end{tikzcd}$$

[– hard to find in literature – no clear source:

– Morse?; J.C. Moore; Samelson?; Hopf?

– stated in Waldhausen paper in 70s.]

Simplicial description of $G^c \times_G EG$:

- start w/ $\underline{EG}$ description. – Milnor join construction;

– consider (top.) category $\mathcal{E}_G$, w/ objects $= G$;

- $Mor(g_1, g_2) = *$ (unique) – h's taking g_1 to g_2: $\underline{g_1 \cdot h = g_2}$.

  so $h = g_1^{-1} g_2$.

- Let $(EG)_\bullet = N_\bullet(\mathcal{E}_G)$. (simpl. space): so

  $(EG)_k$ = k-tuples of comp. morphisms: $(G \times_{\omega} G \times_{\omega} \cdots \times_{\omega} G)$

  $g_1 \xrightarrow{h_1} g_2 \xrightarrow{h_2} \cdots \xrightarrow{h_k} g_{k+1} \cong \underline{G^{k+1}}$

Note: $|EG_\bullet| \simeq * \text{ b/c } \mathcal{E}_G$ has initial objects;

face maps of $EG_\bullet$: $G^{k+1} \xrightarrow{d_i} G^k$ takes

$$d_i(g_0, \cdots, g_k) \Rightarrow \begin{cases} (g_0, \cdots, g_i g_{i+1}, g_k, 1) & i \le k \\ (g_0, \cdots g_{k-1}) & i = k. . \end{cases}$$

s_i – include 1 in ith slot.

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- Show $|EG_\bullet| \simeq$ Milnor's join constr. (exer)

Next, consider simplicial space model for $G^c \times_G EG$:

- $Ad(EG)_k = G^c \times_G (G^{k+1}) \overset{\text{homeo}}{\cong} G^{k+1}$; bdy maps are:

$d_i : G^{k+1} \to G^k$; $\underline{d_0}(g_0, g_1, \dots, g_k) = (g_1^{-1} g_0 g_1, g_2, \dots, g_k)$.)

rest are internal mult or drop off end.:

- $d_i(") = (g_0, g_1, \dots, g_i g_{i+1}, \dots, g_k)$
- $d_k(") = (g_0, g_1, \dots, g_{k-1})$.

- May: writes as $B_*(G^c, G, *)$. in two-sided bar constr.
  $\to$ gives chain cx for $H_*(LBG)$ from G + bdy maps

Waldhausen: construction ('70s) - "cyclic bar construction."

(in fact, top. monoid M suffices for this}, w/ unital, assoc mult $M^{\times 2} \to M$).

- e.g.: $M = \Omega X$ (Moore loops on space X), $\simeq \Omega X$ classical
$= \{ (r, \alpha : [0, r] \to X) : r \geq 0, r \in \mathbb{R}_+, \alpha(0) = \alpha(r) = x_0 \}$.
  - $\times$ by concat. on paths, add on coordinate.)

- define simpl. space $N^{cy}(M)_\bullet$, $N^{cy}_k(M) = M^{k+1}$; w/ bdy maps

- $d_i(m_0, \dots, m_k) := \begin{cases} (m_0, \dots, m_i m_{i+1}, \dots, m_k), & i \leq k-1 \\ (m_k m_0, \dots, m_{k-1}), & i = k. \end{cases}$

- $s_j(-)$ - insert 1's appropriately. - note $\mathbb{Z}/k+1$ acts on $N^{cy}_k(M)$, takes d_i's to other d_j's;

observe: - $\mathbb{Z}/k+1$ does not have same actions.

when M a gp; have simplicial homeo btw $Ad(EG_\bullet) \cong N^{cy}_*(G)$

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need $Ad(EG)_k \overset{d_k}{\longleftrightarrow} N_k^{cy}(G)$:

$$d_k(g_0, \ldots, g_k) = (g_1 \cdots g_k \cdot g_0, g_1, \ldots, g_k)$$

$$(h_k^{-1} \cdots h_1^{-1} h_0, h_1, \ldots, h_k) \longleftarrow (h_0, \ldots, h_k)$$

] ~~dus~~ check these maps: details oh.

• Apply chains to $\underline{N^{cy}(G)}$: – get double complex/chain of chains

$$\cdots \to S_p(G^{k+1}) \xrightarrow{S(\cdot)^{i} d_i} S_p(G^k) \to \cdots$$

$$\big\downarrow d_{simp.} \qquad\qquad \big\downarrow$$

$$\cdots \to S_{p-1}(G^{k+1}) \to S_{p-1}(G^k) \to \cdots$$

Note $S_*(G^{k+1}) \underset{\substack{ch.h.\\ equiv}}{\overset{AW}{\simeq}} S_*(G)^{\otimes k+1}$ (w/ coeffs in field k).

now have $S_*(G)^{\otimes k+1} \xrightarrow{S(\cdot)^{i} d_i} S_*(G)^{\otimes k}$ – gives $HH_*(S_*(G))$

$\underline{\text{Cor}}$: $HH_*(S_*G) \cong H_*(Ad(EG)) \; (= H_*\underline{LBG})$ (Jones '87 result).

$\underline{\text{Note}}$: if G discrete, this is $\underline{HH_*(k[G])}$ ($HH_*(\mathbb{Z}[G])$ is ok too)

$\cong H_*(LBG; k) = \underline{H_*(LK(G,1); k)}$.

– now more general statement – BG not 1-conn here

$\underline{\text{Thm}}$ (Kan - Milnor): for any conn space X, (simplicial) $\exists$ a group $G_X \simeq \Omega X$ (as monoids), so $BG_X \simeq X$.

so $\underline{HH_*(C_*G_X, C_*G_X) \cong H_*LX}$.

$\underline{\text{Recap}}$: 1) $HH_*(C_*\Omega X) \cong H_*(LX)$

2) X 1-conn: $HH_*(C^*X) \cong H^*LX$. – $\underline{\text{relate}}$ the algebras $C_*\Omega X$, C^*X ?

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Q: are the alg. th-theories of these 2 algebras related?

- Blumberg working on this.

Outline of $G^c \times_G EG \simeq LBG$ pf: (Kate's pf). fix $EG \xrightarrow{p} BG$.

Define $\widetilde{LBG} \to LBG$ as $:= \{ \alpha : I \to EG,\ p\alpha(0) = p(\alpha(1)) \}$. - (pts. live in some fiber) so $g(\alpha(1)) = g\alpha(0)g$.

(princ G-bundle)

- this has a free action of G^I by pointwise mult.; $gp^{(?)}$

→ fibration: $G^I \hookrightarrow \widetilde{LBG}$

$\downarrow$

$\widetilde{LBG}/G^I \cong LBG$.

- could mod out by $G \subset G^I$ by constant paths. & $G \simeq G^I$.

so $\widetilde{LBG}/G \simeq LBG = \widetilde{LBG}/G^I$.

now show $\widetilde{LBG}/G \simeq Ad(EG)$: define

$\tilde{\varphi} : \widetilde{LBG} \to G^c \times EG : \alpha \mapsto (g, \alpha(\phi))$, where $\boxed{g \cdot \alpha(1) = \alpha(0)}$ (?) ~~$g\cdot\alpha(1)$~~ ~~$g\cdot\alpha(\phi) = \alpha(0)$~~.

- claim a G-equivariant map; $\tilde{\varphi}(\alpha) = {}^g\alpha(0)$; $\tilde{\varphi}(h\cdot\alpha) = (\alpha(1)\cdot h, \overset{k}{g})$,

where $\alpha(1)hh = \alpha(1)h$ so $k = h^{-1}gh$; (recheck this mult.)

$\alpha(0)_g hh \quad g(0)_g h \to h =$

hence induces fiber bundle map $\widetilde{LBG}/G \to EG \times_G G^c$. under $\tilde{\varphi}$.

claim: $\tilde{\varphi}$ is a h.e.: (so $\widetilde{LBG}/G \simeq EG \times_G G^c$)

observe:

$$\begin{tikzcd} \widetilde{LBG} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \text{p.b.}"] & EG^I \arrow[d, "ev_{0}\times ev_{1}"] \\ EG \times_{BG} EG \arrow[r] & EG \times EG. \end{tikzcd}$$

where

$$\begin{tikzcd} EG \times_{BG} EG \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & EG \arrow[d, "p"] \\ EG \arrow[r, "p"] & BG \end{tikzcd}$$

or

$$\begin{tikzcd} EG \times EG \arrow[r] \arrow[d] & EG \times EG \arrow[d] \\ BG \times BG \arrow[r, "\Delta"] & BG \times BG \end{tikzcd}$$

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$EG \times EG \simeq *$

- note: $BG^I \simeq *$; so $\widetilde{LBG} \xrightarrow{\sim} EG \times_{BG} EG$;

$\underline{\Delta}$: $EG \times_{BG} EG \simeq$ ho fib $(BG \xrightarrow{\Delta} BG \times BG)$

- replace $BG \to BG \times BG$ by fib'n: $\tilde{\Delta}: BG^I \xrightarrow{ev_0 \times ev_1} BG \times BG$

this fiber is $\omega(x_0, x_0) = \Omega BG \simeq G \simeq EG \times G$.

(see that $\tilde{\psi}$ induces that htpy equiv?? - watch unwind details ?)

so $\widetilde{LBG} \xrightarrow{\tilde{\psi}} EG \times G$ is a h.e ; show

Hence: $|N^{cy}_* G| \simeq LBG$. - see S^1-action in LBG;

- see S^1-action in $N^{cy}_* G$: simplicial S^1-action on X_*.

simpl. maps $S^1_n \times X_n \to X_n$ — commuting w/ face & degen ops.

$\mathbb{Z} \simeq \mathbb{Z}/(n+1) = \mathrm{Aut}_{[n]}$ - need switch to t_n-action

→ Defn: A cyclic space/set $X_\cdot$ is a simplicial space/set together w/ operators $\tau_n : X_n \to X_n$ w/: (relations coming from cyclic action on X^{n+1})

1) $\tau_n d_i = d_{i-1} \tau_{n-1}$, $1 \le i \le n$

$\tau_n d_0 = d_n$

2) $\tau_n s_i = s_{i-1} \tau_{n+1}$, $i \le n$

$\tau_n s_0 = s_n \tau_{n+1}^2$

3) $\tau_n^{n+1} = 1$. — check these

→ Thm: (Jones; Dwyer-Hopkins-Kan)? : if $X_\cdot$ is a cyclic space, $|X_\cdot|$ has an S^1-action.

& converse: if X has an S^1-action, $S_*(X)$ is a cyclic set.

(so htpy cats of S^1-spaces & cyclic sets are

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in fact $N^{cy}(G)$ has an S^1-action; $\xrightarrow{\& \simeq}$ LBG an S^1-equiv. map.

Hence, $HH_*(C_*G) = H_*LBG$ should have an "S^1-operator":

$S^1 \times LBG \xrightarrow{\mu_{\text{act}}} LBG$ gives map $H_*(S^1)\otimes H_*LBG \to H_*LBG$:

where $v \overset{\in H_q LBG}{\longmapsto} \mu([S^1]\times \alpha) \in H_{q+1}(LBG)$. (in fact, any LX).

or any S^1-space)

- degree 1-map Δ:

· Jones: · recall $CH_*(A,A)$ has a 'B' operator of degree 1, w/ $B^2=0$.;

in H_*: B ~~[illegible]~~ gives map $HH_*(A,A)\to HH_{*+1}(A,A)$;

& when $A = C_*G$, coincides w/ Δ-op.

other things from Jones: $HC_*(C_*G) = H_*^{S^1}(LBG) = H_*(ES^1 \times_{S^1} LBG)$

- Connes' inverted cyclic homology to study $H_*^{S^1}(X)$, X w/ S^1-action

- spaces of "closed strings" in LX, etc.

$$\mathrm{Emb}(S^1,\mathbb{R}^\infty)\times_{S^1} LX = ES^1\times_{S^1} LX = LX_{hS^1}.$$

(exercise by description of cyclic chain cx)

$$-\quad CC_*(C_*G) = \underset{\substack{\downarrow\simeq\\ C_*LBG}}{CH_*(C_*G)}\ \tilde{\otimes}\ H_*BS^1 \qquad \text{- ass. of due to Brown}$$

$\uparrow$ twisting.

Next: back to TQFTs & S^1-actions there.

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