M283
1/24/08
LN #6

· Conformal Field Theory - relevant in physics, geometry; still topological objects

· Thm: chain-complex valued field theory - related to these conformal theories.

Conformal Cobordism category (Segal) - lectures in mid-'80s;

- Tillmann - organized b-day conference for Segal, forced its ideas were published; publication - ⇝ 2004. (find ref)

- functor: where category accounts for conformal classes of metrics.
  - makes sense for 1+1-dim'l field theory.

Let $M_{g,n}$ be the moduli space of Riemann surfaces Σ of genus g, w/ n disks: (precise def'n to follow) biholomorphic map $\phi: \coprod_{i=1}^{n} D_i^2 \to \Sigma$, where the images of the D^2s are disjoint. ordered.

(more than marked points on surface Σ: .)

- disks have same $\mathbb{C}$ structure as image on Σ.

Hence, $\mathcal{M}_{0,n}$ = space of all biholomorphic maps $\phi: \coprod D^2 \to S^2$, w/ disjoint images / modulo $PSL(2,\mathbb{C})$.

Define the Segal PROP $\mathcal{M}$ w/ obj $\mathcal{M} = \mathbb{Z}^+$, &

$\mathrm{Mor}(n,m) = \mathcal{M}(n,m)$ = mod. space of (possibly disconnected) Riemann surfaces Σ of arbitrary genus, together with biholo. maps $\phi_{in}: \coprod_{i=1}^{n} D_i^2 \to \Sigma$ and $\phi_{out}: \coprod_{i=n+1}^{n+m} D_i^2 \to \Sigma$, w/ disjoint $\phi_*(D_i^2)$'s.

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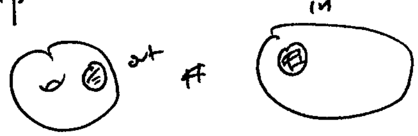
(hence, have $n+m$ embedded p.w. disjoint discs in Σ

Moduli space: can say $\mathcal{M}(m,n) = \coprod_{F} \mathcal{M}(F)$ ↙ topological surface. , F ranges over diffeo. classes of

— Subtleties:

- morphisms spaces are disjoint unions of such subspaces.
- can have multiple components to Σ.

composition = gluing of surfaces; preserves $\mathbb{C}$ structure b/c of biholom. maps relating to std. D^2 $\mathbb{C}$-structures:



: Be careful: delete smaller disk inside; glue on remaining annulus.

→ generates SMC w/ monoidal structure = $\coprod$ of circles/surfaces.

- S_n acts on $\mathcal{M}_{g,n}$ by permuting the order of the disks (hence, acts freely on $\mathcal{M}_{g,n}$); $S_n \times S_m$ acts on $\mathcal{M}(m,n)$.
- $\otimes$: $\mathcal{M}(m_1,n_1) \times \mathcal{M}(m_2,n_2) \to \mathcal{M}(m_1+m_2, n_1+n_2)$ given by disjoint union, as well.

Def'n (Segal): A conformal field theory (CFT) is a functor of $\otimes$ props

$$\mathcal{M} \longrightarrow End(V), \quad V \text{ a vector space.}$$

(perhaps a Hilbert space – careful w/ $\otimes$ then).

(think of as a representation of moduli space)

- no known (explicit) examples of this; can be proved to exist.
- Conjecture (related to TMF): $tmf^0(X) = [X, CFT]$, where CFT = "space of conformal field theories" w/ appropriate topology.
  - i.e., should classify bundles of CFTs.
- recent work by Stölz–Teichner addresses some of this; Pokman Cheung's thesis addresses annular CFT's. (cobordisms are only annuli).

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- pass to chain complexes instead; introduce topological conformal field theories to bridge TQFTs, CFTs: (Manin).

- associate to every homology class or every singular chain in $\mathcal{M}$ a linear map. – get algebro-topological info about $\mathcal{M}$ now.:

Assign to a closed 1-mfld S a $\overset{(\mathbb{Z})}{\wedge}$ chain complex C_S; (over a field k, for convenience).

- Monoidal assignment, so $\exists$ nat. isom $C_{S_1} \otimes C_{S_2} \cong C_{S_1 \sqcup S_2}$

- to a cobordism F between 2 1-manifolds $S_0 \to S_1$, associate a cochain $\mu_F \in C^*(\mathcal{M}(S_0, S_1); \underbrace{\mathrm{Hom}(C_{S_0}, C_{S_1})}_{\uparrow \text{chain cx.}})$, w/ some sort of compatibility (to follow).

literature: μ_F can be a differential form – introduce smooth topology considerations.

- if $|S_0| = p$, $|S_1| = q$, obtain $\mu_F \in C^*(\mathcal{M}(p,q), \mathrm{Hom}(C_{S^1}^{\otimes p}, C_{S^1}^{\otimes q}))$.

compatibility: (precedes 'slick' definitions) – satisfy the following properties:

- $\mu_{n,m}$ denotes $\sum \mu_F$, over all diffeo classes of such cobordisms $n \to m$. (incl. possibly disconnected ones);

- let $\mu: \mathcal{M}(n,m) \times \mathcal{M}(m,p) \to \mathcal{M}(n,p)$ denote gluing map;

so compatibility relation is:

$$C^*(\mathcal{M}(n,p), \mathrm{Hom}(C^{\otimes n}, C^{\otimes p})) \xrightarrow{\mu^*} C^*(\mathcal{M}(n,m) \times \mathcal{M}(m,p), \mathrm{Hom}(C^{\otimes n}, C^{\otimes p})).$$

$\mu_{n,p} \in C^*(\mathcal{M}(n,p), \mathrm{Hom}(C^{\otimes n}, C^{\otimes p}))$

cross. prod. $\oplus$ comp on Hom.

$$C^*(\mathcal{M}(n,m); \mathrm{Hom}(C^{\otimes n}, C^{\otimes m})) \otimes C^*(\mathcal{M}(m,p), \mathrm{Hom}(C^{\otimes m}, C^{\otimes p}))$$

$(\mu_{n,m} \otimes \mu_{m,p})$

– images of $\mu_{n,p}$, $\mu_{n,m} \otimes \mu_{m,p}$ must agree for all n, m, p.

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adjoint rephrasing: define a new PROP $C_*(\mathcal{M})$ ~ S_j ~ $\mathbb{Z}^+$;

w/ $\mathcal{M}\mathcal{N}(n,m) = C_*(\mathcal{M}(n,m))$. (coeffs again in field k).

→ obtain ⊗ by composition then gluing, etc.

Note ~~[illegible]~~ A TCFT is a morphism of PROPs $C_*(\mathcal{M}) \xrightarrow{E} \mathrm{End}(C_*)$
↑ fixed chain cx.

Then $E(1) = C_*$, etc. so C_* a DGA from , etc.

- for each p, q, have $C_*\mathcal{M}(p,q) \otimes C_*^{\otimes p} \to C_*^{\otimes q}$ (eval map);

from cochain description. – family of homomorphisms indexed by chains on $\mathcal{M}$.

– generalized rep'n of chains;

apply homology: have map $H_*\mathcal{M}(p,q) \otimes H_*(C_*)^{\otimes p} \to H_*(C_*)^{\otimes q}$.

– consider $H_0\mathcal{M}(p,q)$ maps = gen by path components of $\mathcal{M}(p,q)$ = diffeo classes of such cobordisms.

– hence obtain linear map for each diffeo class of cobordisms and so yields a TQFT / FA (graded case). – here $V_* = H_*(C_*)$.

so a TCFT reduces to a TQFT in dim 0 after passing to H_*.

Analyze chains in moduli space: Let F_g be a fixed smooth surface of genus g. Recall classical Riemann surface theory:  F_g.

$\mathcal{M}_{g,n} = \{(J, \phi) : \phi : \coprod_n D_i^2 \to F_g$ smooth embeddings;
J a complex structure on $F_g \setminus \mathrm{im}\,\phi\} / \mathrm{Diff}^+(F_g)$

– why is this the same: smooth embeddings define cx structures on $\phi(D^2)$s;
J = – extension gives cx structure outside – rest of F_g.

D^2 open or closed?

* Thm is Riemannian geometry that almost cx structures are intrinsically cx structures on R. surfaces.

* Notice $\mathrm{Diff}^+(F_g)$ acts transitively on $\mathrm{Emb}(\coprod_n D^2, F)$: so let φ_0 be a fixed embedding $\coprod_n D^2 \to F_g$; ~~so~~ abuse of $F_{g,n} = F_g \setminus \mathrm{im}(\varphi_0)$.
  - $F_{g,n}$ has bdy (not m. conf-sing yet).

So $\mathrm{Diff}(F) \cdot \{(J(F_{g,n}), \varphi_0)\}$ is all of $\{(J,\varphi)\}$ above; moreover, $\mathrm{Stab}(\cdot) = \mathrm{Diff}^+(F_{g,n}, \partial)$. So can describe

$$\mathcal{M}_{g,n} = J(F_{g,n}) / \mathrm{Diff}^+(F_{g,n}, \partial).$$

Thm: (Teichmuller; Nilsson sp(?) - proved it) ~~[illegible]~~, $J(F_{g,n}) \simeq *$ for all g,n; action of $\mathrm{Diff}^+(F_{g,n}, \partial)$ is free if $g \geq 2$, $n > 0$.

* if $n = 0$, stab. gps are all finite.

hence $\mathcal{M}_{g,n} \simeq B\,\mathrm{Diff}^+(F_{g,n}, \partial)$. — $\mathcal{M}$ represents families of R. surfaces.
classifies smooth bundles w/ fibre a surface.

- reinterpret (T)CFT w/ surface bundles now (cf. Lurie lecture):

* define new cobordism 2-cat: (note about John Baez's n-cat. café, physics w.s.). ← link!

(2-)PROP Cob_2 à $\mathrm{obj} = \mathbb{Z}^+$;
category $\mathrm{Mor}(n,m)$ = : (idea of Grothendieck → stacks in AG → cl. spaces in AT.)
  * 1-morphisms = cobordisms $\coprod_n S^1 \to \coprod_m S^1$ (or -ples) (not diffeo classes); ← not a set, too big.
  * w/ (2-)morphisms = diffeos of cobordisms.

- preserve automorphisms of ~~[illegible]~~ cobordisms.

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• 2-cat struct on $\mathrm{Vect}_{\mathbb{C}}$: let C_* be a chain cx / k field;

define ~~...~~ 2-PROP $\mathrm{End}(C_*)$: obj = $\mathbb{Z}^+$;

1-Mor (n, m) = $\underline{\mathrm{Hom}}(C_*^{\otimes n}, C_*^{\otimes m})$; chain maps

2-Mor : chain homotopies of chain maps. $n \overset{f}{\underset{g}{\Downarrow}} m$. ← set n / 2-cell n to pt?

(Lurie: calls this an extended TQFT; much like a TCFT :)

- So $\mathrm{Mor}(n, m)$ a cat; so has geometric realization;

Morava: 2-cat $\xrightarrow{B(-)}$ topological cat. so $B(\mathrm{Mor}(n,m)) = \coprod_{[F]} B\,\mathrm{Diff}^+(F)$
↖ $n \to m$ cobs.

Q: how to make $\mathrm{Mor}(n,m)$ small;

— take equivalent small cat? - skeleton category is small.)

so $B(\mathrm{Skel}(\mathrm{Mor}(n,m)) \simeq \coprod_F B\,\mathrm{Diff}(-)$.

• functors changed, though, even though cats equivalent.

• "Meta-theorem" (physics lit - no proof) (Ginzburg - Kapranov, Tradler, Getzler)

- a sum at sketch? pt?

if $\mathcal{E}$ is a chain-cx valued TQFT, (e.g. TCFT, or Ex. TQFT, or others).

then $C_* = \mathcal{E}(S^1)$ is a DGA, chain-htpy comm,

Hochschild cohom
& $HH^*(\mathcal{E}(S^1))$ is self dual, and is a Batalin-Vilkovisky (BV) algebra:

(should say $HH^*(C^*M)$ is a BV-alg: verified (Chas-Sullivan, Tradler).

~~and~~ & $HH_*(C^*M) \cong H_*LM$ (M 1-conn).

Defn: A BV alg: A is a pair (A, Δ), where A is a graded-comm alg,

& Δ is an operator of degree 1 on A w/:

1) $\Delta^2 = 0$,

2) the derivation $\{\alpha, \theta\} = (-1)^{|\alpha|}\Delta(\alpha\cdot\theta) - (-1)^{|\alpha|}\Delta(\alpha)\cdot\theta$ is a derivation
$- \alpha\cdot\Delta(\theta)$. in each variable.

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Chas-Sullivan: (+ Getzler) $\{,\}$ satisfies the graded Jacobi identity

for a graded Lie alg. (of degree 1?)

- such a gr comm alg w/ a Lie bracket $[,]$ s.t. $[,]$ is a derivation in each var is called a Gerstenhaber algebra. (actually, known by Samelson:)

  Ex: $A = H_*(\Omega^2 X)$ has a Lie bracket (Samelson bracket) w/ these properties.

so roughly speaking, $H_*(TCFT)$ has this structure.

- work w/ this algebraic Cat.