M283
1/22/08
LN #4

Last time: 2D TQFTs & FA's (over $\mathbb{C}$)

Recall from last time: equivalence / isomorphism

cat. of TQFTs $\longleftrightarrow$ cat. of Frob. algebras

(w/ morphisms = monoidal natural transformations that are isomorphisms.) (w/ morphisms = isomorphisms of FAs)

Note: More explicitly, morphism of TQFTs = collection of linear isomorphisms $A \to A'$

$\Phi_n = \Phi_1^{\otimes n} : A^{\otimes n} \to (A')^{\otimes n}$ (using $A(\amalg S^1) \cong A(S^1)^{\otimes n}$ isom)

+ cobordism $\Sigma : \underline{n} \to \underline{m}$ $\mapsto$

$$\begin{array}{ccc} A^{\otimes n} & \xrightarrow{\Phi_n} & (A')^{\otimes n} \\ {\scriptstyle A(\Sigma)}\downarrow & & \downarrow{\scriptstyle A'(\Sigma)} \\ A^{\otimes m} & \xrightarrow{\Phi_m} & (A')^{\otimes m} \end{array}$$

commuting (i.e., this is naturality)

why only isomorphisms in Frob. alg. cat?

- any map of Frobenius algebras $A \xrightarrow{\phi} A'$ is an isomorphism:
  - preserves inner product $\theta \circ m$ — nondegenerate so ϕ preserves induced isoms $A \xrightarrow{\nu} A^*$, $A' \xrightarrow{\nu'} (A')^*$ (adjoints of $\langle -,- \rangle$ pairings)
  - so

$$\begin{array}{ccc} A & \xrightarrow{\phi} & A' \\ \cong\downarrow{\scriptstyle \nu} & ? & \downarrow{\scriptstyle \nu'}\cong \\ A^* & \xleftarrow{\phi^*} & (A')^* \end{array}$$

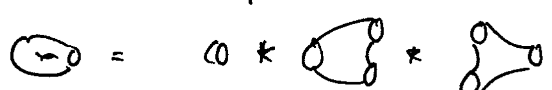
commutes; hence, ϕ has an inverse $\underline{\nu^{-1} \circ \phi^* \circ \nu'}$.

$\Rightarrow$ isomorphism.

- clear in Abrams paper why these categories are equivalent;

Properties from Thursday: E a TQFT;

Prop: Let $\alpha \in A$ correspond to $E($  $): \mathbb{C} \to A$.

Note: 

so $\alpha = \mathbb{C} \to A \xrightarrow{\psi} A \otimes A \xrightarrow{m} A$

$1 \mapsto 1_A \longrightarrow \sum a_i \otimes b_i \mapsto \sum a_i b_i$.

JAN 22 2008

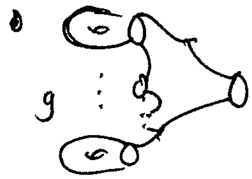
Note: alternate def'n of an FA:

- FA the same as the data:
  - f.d. algebra A over $\mathbb{C}$;
  - coalgebra structure on A, such that $\Delta: A \to A \otimes A$ is a map of A-bimodules

(in fact, dual to multiplication on A).

& counit in coalg structure $= \theta$ (trace map).

1). if $\varphi_g \in \mathbb{C}$ corresponds to $E(\Sigma_g): \mathbb{C} \to \mathbb{C}$, $\varphi_g = \theta(\alpha^g)$.

why? $\alpha^g =$  , $* \; \underset{(\theta)}{\circ} = \Sigma_g$. ✓.

2) If $\{e_i\}$ a basis for A, $\{e_i^*\}$ its dual basis wrt $\langle -, - \rangle$,

(so $\langle e_i, e_j^* \rangle = \delta_{ij}$), then $\alpha = \sum_i e_i e_i^*$. coproduct.

Pf: recall class  $: \mathbb{C} \to A \otimes A$; $= \sum e_i \otimes e_i^* = \varphi(1_A)$

b/c adjoint to $id_s : A \to A$.

hence applying  $\mapsto \mu$ gives $\alpha = \sum_i e_i e_i^*$.

hence $\theta(\alpha) = \dim_{\mathbb{C}} A$, as well.

3) Consider regular rep'n given by $A \xrightarrow{\rho} \mathrm{End}_{\mathbb{C}}(A)$ w/ $\rho(a) = L_a$, $L_a(b) = a \cdot b$,

← matrix for $\rho(a)$ wrt. some basis.

then $\theta(a\alpha) = \text{trace}(\rho(a))$:

4) A is semi-simple as an algebra (i.e., A is a direct sum of 1-dim'l A-modules $\bigoplus \mathbb{C}_\beta$) iff $\alpha \in A$ is invertible.

(recall that successively gluing by  like localizing wrt. multiplication by α; if α invertible, localization has no change on A, - so can consider TQFTs on high-genus surfaces)

JAN 22 2009

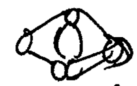
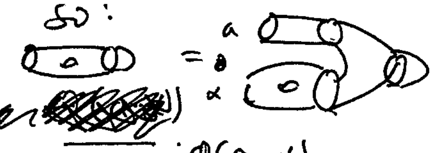
idea: in a semisimple field theory, Theory determined by effect on genus 0 or 1 — key to a lot of current work in the field.

Pf (3): $A \xrightarrow{\rho} \mathrm{End}(A) \xleftarrow{\cong} A^*\otimes A \xleftarrow{\nu\otimes *} A\otimes A$ defining map $\psi: A \to A\otimes A$

— in fact, this is the coproduct on A.

Trace map: $\mathrm{End}(A) \to \mathbb{C}$ is eval $: A^*\otimes A \to \mathbb{C}$.;

but $\mathrm{eval}\circ(\nu\otimes 1) = \langle -,-\rangle = \theta\circ m$. So:

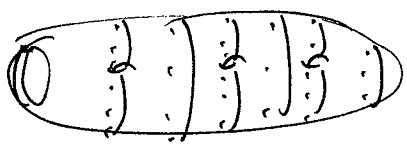
$tr(\rho(a)) = \theta\circ m\circ\psi(a) =$  $=$  $= \theta(a\cdot\alpha)$.

$a \mapsto \theta(\psi\circ m(a))$

Note: confusion in literature (even) about θ — $\theta(a\alpha)$ is trace(a); not just $\theta(a)$.

Aside: other preliminary properties before #4!

Prop: if A is an F.A., replace θ by $\lambda\theta$, $\lambda\in\mathbb{C}^*$, then:
ψ_g changes to $\lambda^{1-g}\psi_g$. (note $1-g = \chi(\Sigma_g)/2$).

Pf: $\Sigma_g =$  g-many copies ... so $\psi_g = \theta\circ m\circ\psi\circ m\circ\psi\circ m\circ\psi\circ\eta = \theta\circ(m\circ\psi)^{\circ g}\circ\eta$

$(g=3)$

Notice: m unchanged by θ-rescaling; $(\theta\mapsto\theta_\lambda = \lambda\theta)$

ψ does change to ψ_λ — compute:

characterize coproduct:

$$\begin{array}{ccc} A & \xrightarrow{\psi} & A\otimes A \\ \downarrow\nu & \cong & \uparrow \nu^{-1}\otimes\nu^{-1} \\ A^* & \xrightarrow{m^*} & A^*\otimes A^* \end{array}$$

— so depends on ν; ν is adjoint to $\langle -,-\rangle : A\otimes A\to\mathbb{C} = \theta\circ m$.

$\nu(a)(b) = \theta(a\cdot b)$, so $\nu_\lambda(a)(b) = \theta_\lambda(a\cdot b) = \lambda\theta(a\cdot b)$;

hence, $\nu_\lambda = \lambda\nu$. so $\nu_\lambda^{-1} = \lambda^{-1}\nu$.; hence,

$\psi_\lambda = \lambda^{-1}\psi$, so $(\psi_g)_\lambda = (\lambda^{-1})^g\cdot\lambda\cdot\psi_g = \lambda^{1-g}\cdot\psi_g$ ✓.

Recall Dijkgraaf-Witten toy model: in 2-D - express as an F.A.:

- G a finite group, $A(S^1) = \mathbb{C}^{P_{S^1}}$, P_{S^1} = iso classes of prin G-bundles on S^1.

- recall $P_X = [X, BG]$ (free classes) $= \pi_0\, \mathrm{Map}(X, BG)$.

so $P_{S^1} = \pi_0 LBG$; hence $A = H^0(LBG; \mathbb{C})$. - relates to string top.

---

multiplication structure: $P_{S^1} \xleftarrow{p_{in}} P_{8} \xrightarrow{p_{out}} P_{S^1} \times P_{S^1}$ ("correspondence diagram").

$\circ$ in out $\circ \circ$

Note $P_{pants} = P_{\infty}$ by htpy equivalence; $\xrightarrow{p_{out}} P_{top\,0} \times P_{bottom\,0}$

w/ $P_{\infty} \xrightarrow{p_{in}} P_{S^1}$ = outer ∞.

take 'cohomology' (mapping spaces: apply $\mathbb{C}^{-}$ to sets.: obtain:

$$\mathbb{C}^{P_{S^1}} \xrightarrow{p_{in}^*} \mathbb{C}^{P_8} \xleftarrow{p_{out}^*} \mathbb{C}^{P_{S^1}} \otimes \mathbb{C}^{P_{S^1}}$$

$p_{in}^!$ (unkeener) ; composite $p_{in}^! \circ p_{out}^*$ gives $m: A \otimes A \to A$.

explicit form for $p_{in}^!$: $\quad p_{in}(\phi)([\gamma]) = \sum \frac{1}{|\mathrm{Aut}\,\beta|} \phi([\beta])$ (fn, bundle on S^1) (classes $[\beta]$ of bundles on 8 restricting to $[\gamma]$ on 'outer' circle)

reformulate iso classes of bundles on 8:

$P_{S^1} = \pi_0 LBG$; $G \to LBG \xrightarrow{\mathrm{Fib}} BG$ (connected) } gives LES in π_*:

$G = \pi_1 BG \to$ so $G \twoheadrightarrow \pi_0 LBG = P_{S^1} \to *$.

$\uparrow$ acts on G. - action is by conjugation : $\pi_0 LBG$ free htpy classes $[S^1, BG]$;

$\pi_1 BG = \pi_0 \Omega BG$ - acts by conjugating by paths.

so: claim $\pi_0 LBG \cong G/\text{conj classes}$.

Alternately: $LBG \simeq EG \times_G G^{conj}$ - can see path components when G discrete.

28 TFT

JAN 22 2008

$$\varphi_8 = \pi_0(\mathrm{Map}(8, BG)) \qquad \begin{array}{c} \mathrm{Map}_\partial(8,BG) \\ \cong \\ \Omega BG \times \Omega BG \end{array} \hookrightarrow \mathrm{Map}(8; BG) \xrightarrow{\text{eval at } * \text{ pt}} BG$$

hence, $\pi_0(\text{fibre}) = G \times G$; so $\pi_0(\mathrm{Map}(8, BG)) = G\times G / \text{action } \pi_1 BG = G$;

– G acts diagonally by conjugation on $G \times G \to \varphi_8 = G\times G/\text{diag conj}$.

Exercise: $\mathrm{Map}(8, BG) = EG \times_G (G\times G)^{\Delta \text{ conj}}$. (see next week, too).

return to picture: $G/\text{conj} \leftarrow G\times G/\text{conj} \xrightarrow{\text{pant}} G/\text{conj} \times G/\text{conj}$

$[g \cdot h] \leftarrow [g, h] \mapsto [g], [h]$

conj-equivariant.

→ hence algebra $A = \mathbb{C}^{G/\text{conj}} \subset \mathbb{C}^G$ – class functions of G.

→ mult. structure is induced: $(d_1, d_2) \mapsto m(d_1, d_2)$ by:

$$m(d_1, d_2)[g] = \sum_{\substack{[g_1, g_2] \in G\times G/\text{conj} \\ \text{s.t. } [g_1 \cdot g_2] = [g] \text{ in } G/\text{conj}}} d_1(g_1)\cdot d_2(g_2)$$

(double coset formula?)
– like a transfer map

Dual multiplication: $(\mathbb{C}^{G/\text{conj}})^* \subset \mathbb{C}^G$; $(\mathbb{C}^G)^* \cong \mathbb{C}[G]$ gp. ring.

claim: $(\mathbb{C}^G)^* \cong \sum z_g g \in \mathbb{C}[G]$ s.t. z_g are constant on conjugacy classes $\equiv Z(\mathbb{C}[G])$. (basic rep'n theory fact).

Hence, dual $A^* = Z(\mathbb{C}[G])$ w/ induced multiplication from gp. ring.

– to get this: consider $((\text{pant}^* v)_!$ – should be on pant instead.

· Next time: return to #4. proof – semisimplicity

· next few weeks: · FAs w/ values in chain complexes, not just vector spaces.
(ST, SFT, floer theory, etc.)

29 TFT

· Then: graded TOFT → (not my) DGA A-ins'; $HH_*(A)$ has a BV-alg. JAN 31 2008

(Costello) Ex: $C^*(M)$ – fund. example of a TOFT.