M283
1/17/08
LN 3

Idea: a TQFT is a functor E: $Cob_n \to$ (Linear Cat)
(say, $\mathbb{C}$-vector spaces)

Cob_n a cobordism category of $n-1$, n mflds;

· taking $\amalg$ to $\otimes$. (not made rigorous in literature, largely)

– we'll want some freedom in terms of cob, linear cats

Recall Atiyah-Segal program: X^{n-1}, two kinds of morphisms:

. diffeos $X_1 \xrightarrow{d} X_2$ & cobordisms $Y: X_1 \to X_3$ – not independent.

→ 'double' category $\mathcal{C}$: 4 types of things:

- objects:
- 2 types of morphisms: horizontal & vertical
  $Mor_{horiz}(a,b)$, $Mor\,vert\,(c,d)$ – each making $\mathcal{C}$ a category
  - obtain $\mathcal{C}_h$, $\mathcal{C}_v$ categories
- picture:

$$\begin{tikzcd} a \arrow[r, "x_1"] \arrow[d, "d_1"'] \arrow[dr, phantom, "A"] & b \arrow[d, "d_2"] \\ c \arrow[r, "x_2"] & d \end{tikzcd}$$

– want something in the middle: A
– '2-morphism'.
– 'source' & 'target' for these?

for $A \in Mor_2$, have the 4 sides (w/ compatibility):

$\sigma_h(A)$, $\tau_h(A) \in Mor\,\mathcal{C}_h$; & $\sigma_v(A)$, $\tau_v(A) \in Mor\,\mathcal{C}_v$.

2 different compositions; (horiz & vertical) $\circ_h$, $\circ_v$.

$\boxed{B} \circ_h \boxed{A} = \boxed{A}\boxed{B}$ or $\boxed{B} \circ_v \boxed{A} = \begin{array}{|c|}\hline A \\ \hline B \\ \hline\end{array}$. – obvious identities for this composition.

(reference: Mac Lane, CW text) – orig due to Ehresmann, 40's.

example: category of sets: w/ set maps, comm diagrams of maps of sets.

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⊛ different from a ~~[illegible]~~ 2-cat – double cat w/ $\mathcal{C}_h \overset{(say)}{=} 1_{\mathcal{C}} = 2$ cat

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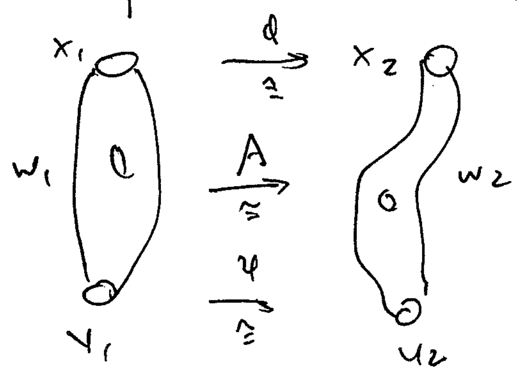
Cobordism category above also an example of such a category:

- objects: closed (oriented) $n-1$-mflds X
- horizontal morphisms: $\mathrm{Mor}_h(X_1, X_2)$ - orientation-pres diffeos $X_1 \to X_2$.
- vert. morphisms: $\mathrm{Mor}_v(X, Y)$ - oriented n-mflds W, w/ orientation-pres. diffeos $\partial W \to X \amalg \bar{Y}$ ← opp. orientation;

(really want: local diffeo $U(\partial W) \to X \times [0,\varepsilon) \amalg (1-\varepsilon, 1] \times Y$
nbhd of bdy — collar structure on W.)

so composition is gluing (now well-behaved smoothly w/ collars)

- 2-morphisms: diffeomorphisms of cobordisms:



, A compat. w/ φ, ψ - diffeos.

— compose by fn. composition (horiz) or by gluing (vertical).

Linear double category: $\mathrm{Vect}_{\mathbb{C}}$

objects: vector spaces / $\mathbb{C}$;

morphisms:
- Mor_{horiz} = isomorphisms $V \to V'$
- Mor_{vert} = linear maps $V_1 \to W$.
- 2-morphisms: commuting squares: if

$$\begin{tikzcd} V_1 \arrow[r, "\varphi", "\cong"'] \arrow[d, "\phi_1"'] & V_2 \arrow[d, "\phi_2"] \\ W_1 \arrow[r, "\varphi'"'] & W_2 \end{tikzcd}$$

commutes, have a single 2-morphism A here.

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Then a TQFT is a functor of 2-categories $E : \mathrm{Cob}_n \to \mathrm{Vect}\,\mathbb{C}$ taking $\sqcup$ to $\otimes$. (i.e., a monoidal functor).

- since there are few interesting 2-morphisms in $\mathrm{Vect}_{\mathbb{C}}$, LT assoc to cobordism depends only on (diffeo). type of cobordism.
- introduce more interesting structures later in course.

Defn: A symmetric monoidal category (SMC) $\mathcal{C}$ has a functor $\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$, and functorial isoms (Nat xforms)

$$\alpha_{x,y,z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z,$$

$$\tau_{x,y} : X \otimes Y \to Y \otimes X$$

satisfying relations.

- 1) $\tau^2 = id$; 2) (Stasheff pentagon) $\alpha_{x\otimes y,z,w}\ \alpha_{x,y,z\otimes w} = (\alpha_{x,y,z} \circ id_w) \circ \alpha_{x,y\otimes z,w} \circ (id_x \otimes \alpha_{y,z,w})$

(draw out).

3) $(\tau_{x,y} \otimes id_z) \circ \alpha_{x,z,y} \circ (id_x \otimes \tau_{y,z}) = \alpha_{z,x,y}\ \tau_{x\otimes y,z}\ \alpha_{x,y,z}$.

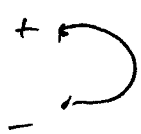
- coherence diagrams.

a symmetric monoidal functor preserves $\alpha, \tau, \otimes$. - a TQFT is such a functor

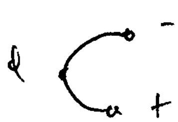
1-Dimensional TQFTS: • objects - oriented 0-mfld - signed pts.

- morphisms: 1-dim'l oriented cobordisms btw pts:

So: assign V to $\bullet^+$; so $V^* = \bullet^-$;

- cobordisms:  gives duality $V \otimes V^* \to \mathbb{C}$;

 $= id_V$;

&  is $\mathbb{C} \to V^* \otimes V = \mathrm{End}(V)$

$1 \mapsto id_V$.

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&  $\mapsto \dim_{\mathbb{C}} V$.

hence data of V determines the E: so $\exists$ equivalence of cats.:

· 1-dim'l TQFTs $\Longleftrightarrow$ finite dim'l $\mathbb{C}$-vector spaces.

2-dim'l TQFTs: · Cob_2 · any closed 1-mfld diffeo to $\coprod_k S^1$.

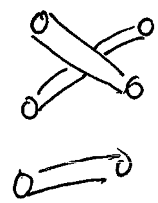
– reduce data to these objects: fix S^1, take k copies.

– i.e., · objects are $\mathbb{Z}^+$ (= # of S^1's)

· $Mor(n,m)$ = cobordisms Σ with $n+m$ bdy components w/ parametrizations of bdy $\partial_{in}\Sigma \xrightarrow{\cong} \coprod_n S^1$, & $\partial_{out}\Sigma \xrightarrow{\cong} \coprod_m S^1$ (avoiding collar specifics)

– all mod $Diff^+(-,\partial)$. – take diffeo classes of such cobordisms

– lost some data; but value on cobordism depends only on diffeo type (will not do this at some later date)

Then: · $\exists$ maps $\Sigma_n \to Mor(n,n)$:  permutations.

· take v.s. V; define $End(V)$ category:

· objects = $\mathbb{Z}^+$; $mor(p,q) = Hom_{\mathbb{C}}(V^{\otimes p}, V^{\otimes q})$.

– both symm. monoidal cats: $\otimes = +$ on objects, $\amalg$ or $\otimes$ on morphisms.

Defn (MacLane): A PRO is a SMC $\mathcal{C}$ whose objects are $\mathbb{Z}^+$, where $\otimes$ on objects is addition.

A PROP is a PRO with permutations: i.e., $\exists$ a map $S_n \xrightarrow{\ell_n} \mathrm{Mor}(n,n)$

of monoids compatible w/ SMC structure:

- 1) if $\sigma_1 \in S_m$, $\sigma_2 \in S_n$, $(\sigma_1 + \sigma_2) \in S_m \times S_n \subset S_{m+n}$.

  so $\ell(\sigma_1 + \sigma_2) = \ell(\sigma_1) \otimes \ell(\sigma_2)$.

2) if $\tau_{m,n} \in S_{m+n}$ ~~...~~ exchanges 1st m, last n 'letters',

  then $\ell(\tau_{m,n}) = \tau_{\underline{m},\underline{n}}$ in $\mathcal{C}$.

So a 2D TQFT is a functor of PROPs from $\mathcal{C}ob_2$ to ~~...~~ End(V), some V.

Classification: (thm - folk thm) - Abrams article a 2DTQFT $\Rightarrow$ the same as giving a f.d. assoc. $\mathbb{C}$-algebra A, commutative, with unit, w/ a linear map $\theta: A \to \mathbb{C}$ s.t. $\langle x,y \rangle := \theta(x \cdot y)$ is a nondegen, bilinear form. (these are called Frobenius algebras)

(Note: this will not suffice to describe our TQFTs: - need grading)

- nat: $A = H_*(M^n, k)$ ~~...~~ a graded Frob. algebra over k:
  - $A \otimes A \to A$ is the intersection product &
  - $H_*(M^n, k) \xrightarrow{\theta} k$ is the projection to $H_0(M, k)$.

    (M connected). - maps in cohomology π to $H^n(M, k)$.)

  so PD $\Leftrightarrow$ nondegeneracy of θ. (spectrally)

- shift grading: $A_q = H_{q+n}(M)$. $(= H_*(M^{-TM}))$

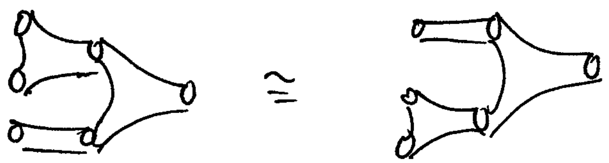
- recall: $S^1 \times S^1$ should give $\dim_{\mathbb{C}} A$; in graded setting, get alt. sum of dim's, hence Euler characteristic if $A = H_* M$.)

Abrams: proves that the category of 2D TQFTs (where the morphisms are monoidal nat. transfs) is isomorphic to cat. of Frob. algebras, 22 TFT

where the morphisms are ISOmorphisms of FA's, Note on how these relate: $E : \mathrm{Cob}_2 \to \mathrm{End}_V$, then V is an FA.

Yes $V = E(1)$. ; $\mu : V \otimes V \to V$ given by $E(\ )$;

- commutative: need $E(\ \circ\ ) = E(\ )$ — but these are diffeomorphic rel ∂. ✓

- assoc: 

- θ : given by $E(D^2)$: — $V \to \mathbb{C}$.

- unit: $E(\ ) : \mathbb{C} \to V$. — nondegeneracy comes from Segal S - composition / duality proposition.

  (coproduct later)

- other features:

__Prop__: 1) $\exists$ a distinguished elt $\alpha \in A$ corresponding to a

(image of $\mathbb{C} \xrightarrow{E(\ )} V$) : if $\theta\phi_g$ = invariant for (genus) $\in \mathbb{C}$,

then $\underline{\phi_g = \theta(\alpha^g)}$. (picture proof).

2) if $\{e_i\}$ a $\mathbb{C}$-basis for V, & $\{e_i^*\}$ its dual basis wrt. $\langle\ ,\ \rangle$,

then $\underline{\alpha = \sum e_i \cdot e_i^*}$ (really, already proved this earlier)

3) regular rep'n $\rho : V \to \mathrm{End}(V)$ induced by left. mult.
$a \mapsto a \cdot -$

Then ~~...~~ for any $a \in V$, $\theta(a \cdot \alpha) = \mathrm{trace}(\rho(a))$.

(nontrivial, not too hard). — so θ - often called ~~trace~~ map.

Then $\theta(\alpha) = \dim V$. $\checkmark$. $= \mathrm{tr}(\mathrm{id}_V)$

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4) A is semisimple as an alg; (ie A as left A-mod by μ) is isom to direct sum of 1-dim'l A-mods).
if and only if α is invertible in A.

note $\Phi(\quad)$ is mult by α — invertible.

5) if we rescale $\theta : A \to \mathbb{C}$ by λ : we change
$$\varphi_g = \Phi(\Sigma_g) \text{ to } \lambda^{1-g}\varphi_g = \lambda^{\chi(\Sigma_g)/2}\varphi_g.$$

RFs: 3 ; 4 – next week.