M 283
4/15/07
LN #2

Last time: motivation for field theories from physics

Today: actual defn's of structures:

n-dim'l or (n-1) n-dim'l.

Defn (TQFT) - fluid concept - will go through a few stages

(Witten; Atiyah-Segal) - late '80s):

• several parts: I) functor E : $\{$closed, oriented $(n-1)$-mflds + diffeos$\}$ $\longrightarrow$ $\{\mathbb{C}$-v.s.'s & isoms$\}$

(physics interp: $E(X)$ = v.s. of fns / sections of a bundle on a space of fields

Ex: $X = S^1$ (2-dim'l case); space of fields is LM or more generally $\mathrm{Map}(X, M)$,

so $E(S^1) = \Gamma\left(\begin{smallmatrix} \mathcal{L} \\ \downarrow \\ LM \end{smallmatrix}\right)$.)

E satisfies some properties:

• "$E(X_1 \sqcup X_2) = E(X_1) \otimes E(X_2)$": for any finite family $\{X_\alpha\}_{\alpha \in I}$ of $(n-1)$-mflds, $\exists$ M_I :

$$\prod_{\alpha \in I} E(X_\alpha) \longrightarrow E\left(\coprod_{\alpha \in I} X_\alpha\right),$$

multilinear, satisfying universality property of such maps.

Hence establishes $E\left(\coprod_{\alpha \in I} X_\alpha\right) \cong \bigotimes_{\alpha \in I} E(X_\alpha)$.

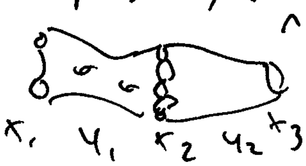
Hence, obtain an S_n-equiv. map $E(X)^{\otimes n} \longrightarrow E\left(\coprod_n X\right)$.

II). assign to an oriented cobordism Y from X_1 to X_2 a linear map

$\Psi_Y : E(X_1) \longrightarrow E(X_2)$ satisfying some properties:

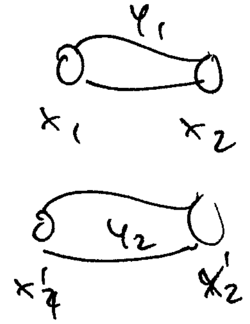
(avoid ultra-careful language for 1st cut).

a) Ψ_Y depends only on oriented diffeo. type of Y; (fixing bdys)

b) 'gluing': 

$\Psi_{Y_1 * Y_2} = \Psi_{Y_2} \circ \Psi_{Y_1}$

c) Tensorial:



Then $\psi_{Y_1 \amalg Y_2}$ "=" $\psi_{Y_1} \otimes \psi_{Y_2}$

$$: E(X_1 \amalg X_1') \to E(X_2 \amalg X_2')$$

$$E(X_1 \amalg X_1') \cong E(X_1) \otimes E(X_1') \qquad E(X_2 \amalg X_2') \cong E(X_2) \otimes E(X_2').$$

III) Extra data: $\psi_{X \times I} = id_{E(X)}$. — cylinder is trivial cobordism.

(Note: can describe this notion via 2-categories - later!)

Note that $\emptyset$ is a closed $n-1$-mfld; claim $E(\emptyset) = \mathbb{C}$:

$$E(X) = E(X \amalg \emptyset) = E(X) \otimes_{\mathbb{C}} E(\emptyset) \quad \text{so} \quad E(\emptyset) = \mathbb{C}.$$

so for Y a closed n-mfld, view as cobordism from $\emptyset$ to $\emptyset$..

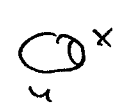
hence obtain $\psi_Y : \mathbb{C} \to \mathbb{C} \iff \psi_Y \in \mathbb{C}$ (invariant for closed n-mflds).

- our perspective: number assigned to closed mflds, w/ something that allows computation from linear operators (arising from pieces).

Questions: 1) Do the numbers ψ_Y, Y closed n-mfld, determine the theory? (if not, what does?)

2) Are there restrictions to the type of diffeomorphism invariants that can arise this way? (ex: Euler char, - is there a TQFT that producing them?)

3). Is $E(X)$, X an $n-1$-mfld, spanned by $\{\psi_Y(1) \in E(X),\ \partial Y = X\}$

(ie,  gives $\psi_Y : \mathbb{C} \to E(X)$), $\mathbb{C} = E(\emptyset)$

4) classify TQFTs - moduli space of them? define isoms of TQFTs;

Exercise: (algebraical problem). Let A be a comm. ring w/ unit, M and N modules over A. These modules are fin. gen, projective modules, and 'in duality' if and only if there exist homs of A-modules $\alpha: A \to M \otimes_A N$, $\beta: N \otimes_A M \to A$,

s.t. the following compositions are id:

(1) $\underset{\underset{A \otimes_A M}{\shortparallel}}{M} \xrightarrow{\alpha \otimes M} M \otimes_A N \otimes_A M \xrightarrow{M \otimes \beta} \underset{\underset{M \otimes_A A}{\shortparallel}}{M}$, $\quad \underset{\underset{N \otimes_A A}{\shortparallel}}{N} \xrightarrow{N \otimes \alpha} N \otimes M \otimes N \xrightarrow{\beta \otimes N} \underset{A \otimes_A N}{N}$.

Pf: in Spanier's topology book (cat of spaces w/ stable maps)

("in duality": means β determines $N \xrightarrow{A\text{-hom}} \mathrm{Hom}_A(M, A)$ - an isom,

and

Relevance: TQFT forces this structure:

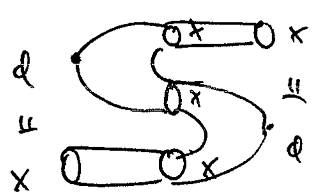
$\cdot$ $U = X \times I$: cobordism from ϕ to $X \amalg \bar{X}$ ← appr. orientations;

so $\alpha = \psi_U : \mathbb{C} \to E(X) \otimes E(\bar{X})$..

Similarly: other cobordism $V = X \times I$: from $X \amalg \bar{X}$ to ϕ.

gives $\beta = \psi_V : E(X) \otimes E(\bar{X}) \to \mathbb{C}$.

why does it satisfy (1)? compute $(1 \otimes \beta) \circ (\alpha \otimes 1)$:



$\cong X \times I : (X \to X)$ - so $(1 \otimes \beta) \circ (\alpha \otimes 1) = \psi_{X \times I} = id_{E(X)}$.

- same for opposite compositions.

hence have.

$\to \bar{\beta}: E(\bar{X}) \xrightarrow{\cong} E(X)^*$ isom. - duality.

what is induced map: $\alpha: \mathbb{C} \to E(X) \otimes E(X)^*$ (using β - isom here).

- adjoint to id: $\in \mathrm{Hom}_{\mathbb{C}}(E(X), E(X)) = E(X) \otimes_{\mathbb{C}} \mathrm{Hom}(E(X), \mathbb{C})$.

- so if $\{b_1, \ldots, b_k\}$ a basis for $E(X)$, $\{b_1^*, \ldots b_k^*\}$ dual basis,

claim $\alpha(1) = \sum_{i=1}^{k} b_i \otimes b_i^*$ - adjoint of id map.

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Similarly: $\beta : E(X)^* \otimes E(X) \to \mathbb{C}$ given by evaluating, adjoint to $\underline{id}$.

Conclude: $\Psi_{X \times S^1} = \underline{\dim E(X)} \overset{\in \mathbb{Z}!}{:}$ constant $\kappa(1)$, character.

Example of a TQFT ($\underline{\text{Dijkgraaf–Witten toy model}}$). TQFT in dim n.

- associated to a finite gp $\underline{G}$. (also generalize to cpt Lie gps). – Freed–Hopkins–Teleman

fix dim $= n$: • 1st describe the invariant Ψ_Y here Y closed n-mfld.

Ψ_Y = weighted # of $\underline{\text{isom}}$ classes of prin. G-bundles over Y.

• for ~~...~~ principal G-bundles $\begin{smallmatrix} P \\ \downarrow \\ Y \end{smallmatrix}$ (reg. cov. space here).

weight $= \dfrac{1}{|\mathrm{Aut}\, P|}$ – so $\Psi_Y = \sum \dfrac{1}{|\mathrm{Aut}\, P|}$, $[P] \in \underline{\{G\text{-bundles}/Y\}/(\text{Isom})}$

↖ automorphisms cover $id: Y \to Y$: $\begin{tikzcd} P \arrow[rr, "\theta"] \arrow[dr, "p"'] & & P \arrow[dl, "p"] \\ & Y & \end{tikzcd}$

– finite here (if Y cpt)

From covering space theory: we know there isom classes:

- $\{G\text{-bundles}/Y\}/\text{Isom} \overset{\cong}{\Longleftrightarrow}$ reps $\mathrm{Hom}(\pi_1 Y, G)/$(G acts by conjugacy).

$= [Y, BG] = [Y, K(G,1)]$ (since G discrete)

– $\underline{\text{free}}$ htpy classes (no b.p.)

• if b.p. $\underline{\text{were}}$ fixed: $\langle Y, BG \rangle$ – would get homs $(\pi_1 Y, G)$ $\underline{\text{not}}$ upto conjugacy.

• $\underline{\text{Given}}$ a rep. $P \to G$, how do we consider $\mathrm{Aut}(P)$?

- P comes from a hom $\rho: \pi_1 Y \to G$ as follows:

$P \cong \tilde{Y} \times_\rho G$: • have $\underline{\text{left}}$ action of $\pi_1 Y$ on $\tilde{Y}$ by deck xforms;

(↑ univ. cover)

so for $\alpha \in \pi_1 Y$, $(y, g) \sim (\alpha y, \rho(\alpha) g)$ ($\underline{\text{identify}}$), all $\alpha \in \pi_1 Y$

$= \tilde{Y} \times G/\sim$

- have $\underline{\text{residual}}$ right G-action here now.

Now, $\underline{\text{Auts of } P}$ lift to $\overset{\pi_1 Y \times G-}{\underline{\text{equiv}}}$ maps : $\tilde{Y} \times G \xrightarrow{\theta} \tilde{Y} \times G$

- $\pi_1 Y$ acts on left, G on right.

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by G - trace ones of $\tilde{Y} \times G$, same as $\pi_1 Y$-equiv maps $\underline{\tilde{Y} \to \tilde{Y} \times G}$.

Then $(y, 1) \mapsto (\tilde{y}', \theta)$, $\theta \in G$, where $\underline{p(y) = p(\tilde{y}')}$ $(p : P \to \tilde{Y})$.

— same fibre in $\tilde{Y}$ - related by deck xfm.

— so can invert. deck xform & get map $\underline{(y, 1)} \to \underline{(y, \theta')}$. - get same $\theta_y \in G$ associated.

⊕ equivariance:

$$\begin{tikzcd} \tilde{Y} \times G \arrow[r, "\theta"] \arrow[d, "\text{act by } \alpha"'] & \tilde{Y} \times G \arrow[d, "\text{act by } \alpha \in \pi_1 Y"] \\ \tilde{Y} \times G \arrow[r, "\theta"] & \tilde{Y} \times G \end{tikzcd} :$$

require θ to commute w/ $p(\alpha)$ - for all $\alpha \in \pi_1 Y$. - θ centralize $\text{im}(p) \subset G$ - hence: $\boxed{\text{Aut}(P) = C_G(p(\pi_1))}$ ⊗.

Interpretation: P = isom class det. by $p : \pi_1 Y \to G$ up to conj. by G;

· ~~[illegible]~~ this orbit $\cong G / Z_G(p(\pi_1 Y))$.

so $\displaystyle\sum_{\substack{[P] \text{ isom} \\ \text{classes of} \\ \text{prin } G\text{-bundles}}} \frac{|G|}{|C_G(\pi_1(Y))|} = |\text{Hom}(\pi_1 Y, G)|$

— pulling out $|G|$: get.

$\displaystyle |G| \sum_{[P]} \frac{1}{|\text{Aut}(P)|} = |\text{Hom}(\pi_1 Y, G)|$; so $\displaystyle \Psi_Y = \frac{|\text{Hom}(\pi_1 Y, G)|}{|G|}$.

our invariant assoc to Y:

vector spaces for $n-1$ mfld, / maps for n-mflds:

$E(X)^{n-1} = \mathbb{C}^{\Phi_X}$ ($=$ isom classes of prin. G-bundles on X) — all fns from these fields

— f. dim'l ~~[illegible]~~ set

— f.d. vector space.

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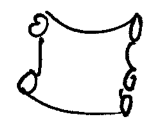
Remaining follows from similar to contra: $\partial Y^n = X$,

take $\psi_Y(c) \in E(X) = \mathbb{C}^{P_X}$ to be: $\psi_Y(P) = \sum_{\substack{[Q] \text{ isom. classes of } G\text{-bundles} \\ \text{over } \underline{Y} \text{ s.t. } \boxed{Q|_{\partial Y} = P}}} \frac{1}{|Aut(Q)|}$.

- analogue of 'path integrals'.
  - 'integrate' over 'fields' (& isoms fix $\underline{Q|_{\partial Y}}$.)
  - only finite! & $\frac{1}{|Aut(Q)|}$ like a transfer map factor.

Different perspective: given cobordism Y: from X_1 to X_2



- restriction maps: $P_{X_1} \xleftarrow{p_{in}} P_Y \xrightarrow{p_{out}} P_{X_2}$
- apply same alg. for (contravariant): $\mathbb{C}^{P_{X_1}} \xrightarrow{p_{in}^*} \mathbb{C}^{P_Y} \xleftarrow{p_{out}^*} \mathbb{C}^{P_{X_2}}$

so the map is $\underline{(p_{out}^*)^! \circ (p_{in}^*)}$. ← not quite defined yet; transfer or shriek map.

- for closed mflds get # of isom classes. → invariant.
- $p_{in}^*: \mathbb{C}^{P_{X_1}} \to \mathbb{C}^{P_Y}$ given by $p_{in}^*(f)(P) := \underline{f(p_{in}(P))}$.
- $(p_{out})^!: \mathbb{C}^{P_Y} \to \mathbb{C}^{P_{X_2}}$: $p_{out}^!(f)\begin{pmatrix} P \\ \downarrow \\ X_2 \end{pmatrix} = \sum_{\substack{[Q] \text{ isom. classes of} \\ \text{bundles on } Y \\ \text{w/ } Q|_{X_2} = P}} f(Q)/|Aut(Q)|$.

Q: study/compute how bundles on X_1 evolve to bundles on X_2 – or fns on such bundles.. – study all bundles on $\underline{Y}$ restricting to X_2, then restrict to $\underline{X_1}$.

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Start more categorical framework: 2-categories: $\mathcal{C}$

- objs $\mathcal{C}$; if a, b are objects, $\mathrm{Mor}(a,b)$ is a category as well:
  - objs $(\mathrm{Mor}(a,b))$ = 1-morphisms;
  - $\mathrm{Mor}($ " $)$ = 2-morphisms. (w/ composition laws & compatibility)

Ex: start w/. a cobordism category; - skip.

Ex: (linear) - objs = $\mathbb{C}$-vector spaces;
- $\mathrm{Mor}(V_1, V_2)$ category:
  - objs = linear transfs $V_1 \to V_2$,
  - Mor $\in$ (L_1, L_2)

$$\begin{tikzcd} V_1 \arrow[r, "L_1"] \arrow[d, "\theta_1"'] \arrow[dr, phantom, "2"] & V_2 \arrow[d, "\theta_2"] \\ V_1 \arrow[r, "L_2"'] & V_2 \end{tikzcd}$$

pairs of isoms making square commute.

(target of a TQFT) - (see Thursday).

Thursday: - classify 2-dim'l $\mathbb{C}$-theories (Frob. algebras) - Costello.
- semisimple algebra: rep decomposes into 0-dim'l pieces.
  $\to$ TFT implications. (Teleman in Graeme's talk thy uses this)

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