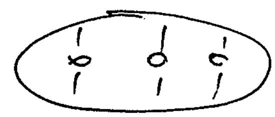
1/10/07
M283
W#1

Topological field Theories

* Josh - area exam next week! *

TQFT: lots of applications to math - invariants: (n+1)

- mid 1980s: Witten (physics); Atiyah-Segal (axiomatics)
  - idea: assemble into an invariants of closed mflds satisfying some gluing properties.
  - assign invts to mflds w/ bdy; assemble/glue to get invt for closed mfld.



Apps: 1. Donaldson-Floer theory, - (3+1 = 4-dim'l theory.)
- invariants to 4-mflds w/ bdy + other data to 3-mflds (= bdy's)

2. Jones, Kauffman polynomial invariants (2+1-dim'l TQFT)
3. Gromov-Witten theory (1+1 - dim'l)
4. (new) Symplectic field theory (arb 2n-dim'l theory)
5. string topology
6. Freed-Hopkins-Teleman - Twisted K-theory.

- highlights some algebra & category theory:
  1) Frobenius Algebras, Gerstenhaber algebras, Batalin-Vilkovisky algebras,
  2) tensor algebras/categories, operads, 'props',

Goal: classification of such theories.

1st wk: classify certain 2-d ones.

Costello: classification of topological conformal field theories

- ~~alg~~ algebra: vector spaces over k, char $k=0$.
  - other examples: need more general algebra - pos. char, htpy action
  - defined on chains / cochains up to htpy.

• recently: "derived" versions of theories: replace algebra w/ stable htpy theory - see lectures by Lurie. (joint w/ Hopkins)

relation to QFTs: (superficial account of nonrigorous physics).

(motivate defns & physically) - refs: Segal, "Topological structures in string theory" (2002) - expository.

19th cent. physics - electromagnetic field strength:
- 2 3-dim'l vector fields E, B in $\mathbb{R}^{3,1}$
  - write as fn $F: \mathbb{R}^4 \to \mathbb{R}^6$ &
- arrange this as a 2-form $F(x) = \sum_{1\le i<j\le 4} F_{ij}(x)\, dx_i \wedge dx_j$ (since $6 = \binom{4}{2}$)

w/ constraint: Maxwell's eqns for em: $dF = 0$ & $d * F = 0$. - (harmonic-form-like constraint.)

cov. ext. deriv, hodge dual

- depends on a metric on $\mathbb{R}^4$

- different fields: gravitational field: replace $\mathbb{R}^4$ w/ X^4, 4-mfld; w/ metric g - on space of metrics on X^4
- consider functional $S(X,g) = \int_X R_g \, d\mathrm{vol}$

R_g scalar curvature of g

gravitational field: view as metric g on X, a critical point of S
- critical eqns = Einstein field eqns.

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(other ref: Feynman's "lectures on QFT")

1950s: if $X \subset \mathbb{R}^4$ is a non simply connected, open set, then one could have an EM field w/ 0 field strength $(F=0)$ but w/ still some physical effect present (Aharonov-Bohm effect).

Chern (et al.): math for EM field should be a connection A on a cx line bundle L over X^4. (possibly trivial) w/ curvature form satisfying Maxwell's eqns

recall connection terminology:

a (principal) connection is a $U(1)$-equiv. splitting of $T(P_L)$ (P_L = prin. $U(1)$ bundle on X assoc to L : $U(1) \to P_L \to X$

$\omega_A : TP_L = p^*TX \oplus T_{vert}P$ — a $U(1)$-equiv splitting
($T_{vert}P$: 1-dim'l bundle)

- then the assoc. covariant derivative $D_A : \Omega^q(X;\mathbb{R}) \to \Omega^{q+1}(X;\mathbb{R})$

linear, satisfying $D_A(f\cdot\sigma) = f\cdot D_A\sigma + df\wedge\sigma$, $f\in C^\infty(X;\mathbb{R})$.

Source of this? ($q=0$:) $\Omega^0(X;L) = C^\infty(X;\mathbb{R}) \to \Omega^1(X;L)$
$= T(L)$ (smooth sections) — coeffs in L now;

$D_A(\sigma)(V) = [\tilde{V}, \sigma]$

$\tilde{V}$: Lie lifting (using splitting) of vector field V.

so: $\Omega^0(X) \xrightarrow{D_A} \Omega^1(X) \xrightarrow{D_A} \Omega^2(X)$, composite F_A, gives the curvature form:

$F_A \in \Omega^2(X;\mathbb{R})$
(or in general $\Omega^2(X; \mathrm{ad}\,P)$ adj bundle of P.)

Properties: $\cdot\ D_A F_A = 0$ (closed condition) – Bianchi ident).

$\cdot\ [F_A] = c_1(L)$ (up to constant multiple) – Chern–Weil thy.

or think of connection as a parallel transport operator:

– $\gamma: I \to X$ curve : obtain linear operator $\tau_A(\gamma): L_{\gamma(0)} \to L_{\gamma(1)}$,

w/: • $\tau_A(\gamma)$ indep of parametrization of γ,

• gluing property : $\overset{\gamma_1 \# \gamma_2}{\underset{\gamma_1 \quad \gamma_2}{\frown\frown}}$: $\tau_A(\gamma_1 \# \gamma_2) \cong \tau_A(\gamma_2) \circ \tau_A(\gamma_1)$.

(– categorical data) – (identity data?)

relate to curvature form:

take $x_0 \in X$, $\tau_0 : \Omega_{x_0} X \longrightarrow \mathrm{Iso}\, L_{x_0} \cong \underline{U(1)}$ taking loop up to mult in $U(1)$.

flat connection: $:\equiv F_A = 0$

$\Rightarrow \tau_A(\gamma)$ depends only on (path) htpy class of γ – τ_A factors through $\pi_1(X, x_0)$.

– gives rep'n $\tau_A : \pi_1(X, x_0) \to U(1)$. (holonomy).

Hence, EM fields w/ 0 field strength physically are understood via rep'n theory of $\pi_1 X \to U(1)$ (or other Lie gps / prin bundles).

relate ‖ xport to field strength: suppose Σ a surface w/ $\partial\Sigma = S^1$. w/ $\sigma: \Sigma \to X$, $\gamma := \partial\sigma : S^1 \to X$ (based at x_0).

define $F(\sigma) = \int_\Sigma \sigma^* F_A$; then $e^{2\pi i F(\sigma)} = \tau_A(\gamma) \in U(1)$.
(up to factor?)

hence if $F(\gamma)=0$, $\tau_A(\gamma)=1$ — hol. map'n is constant.
– b/c γ is 0 in $H_1(X) = \pi_1(X)^{ab}$.

Quantization of such models: – generalize:

A string field (B-field, or gerbe, gerbe w/ connection)

[1st: return to connection case:
conn A on $L \to X^4$:
have functor: $\tau_A : \mathcal{P}_X \to \mathcal{L}$ ((cats) P_X, $\mathcal{L}$ top. cats)
(path cat) (lines)

• ob $\mathcal{P}_X$ = pts in X; mor $\mathcal{P}_X(x,y) = \{\text{path } \gamma : [0,t] \to X\}$
$\gamma(0)=x,\ \gamma(t)=y$.

– turn category through concat. (need t variable – Moore paths).

• ob $\mathcal{L}$ = complex lines $\subset \mathbb{C}^\infty$; $= G_1(\mathbb{C}^\infty)$.

mor(L_0, L_1) = isoms. of lines.]

– Same setup in string field theory:

• associate a $\mathbb{C}$-line L_γ to every loop $\gamma : S^1 \to X$ (indep of param).

– consider image of loop:

– Space of closed strings $LX // S^1$ ($= U(1)$ $\cong SO(2)$) $= \{ S \subset \mathbb{R}^\infty : S$ closed, or. 1-mfld ($\cong \amalg S^1$) w/ map $f : S \to X. \}$
↑ synchronization

– topologize as follows: (GMTW, Thm):
– $\mathrm{Emb}(\amalg_k S^1, \mathbb{R}^\infty) \times$ Map$(\amalg_k S^1, X)$
($\simeq$ w/ free action of Diff $\mathrm{Diff}^+(\amalg_k S^1)$

– gives model for homotopy orbit space
$E\mathrm{Diff} \times_{\mathrm{Diff}}$ Map(S,X)

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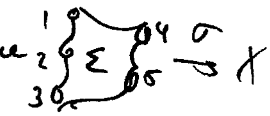
$= E\mathrm{Diff}^+ \times_{\mathrm{Diff}^+} \mathrm{Map}(\coprod_k S^1, X)$ – write as $LX//S^1$ – stacky notation.

– hence: have linebundle over $LX//S^1$ string space;

connection: to a cylinder $\xrightarrow{\gamma} X$ (curve of loops), want parallel transport:

$B_\gamma : L_{\gamma(0)} \to L_{\gamma(1)}$; more generally: for any surface



$\xrightarrow{\sigma} X$

w/ $\Sigma \subset \mathbb{R}^d \times [0,t]$, assign $B_\sigma : L_{\partial\sigma_1} \otimes L_{\partial\sigma_2} \otimes \cdots \otimes L_{\partial\sigma_3} \to L_{\partial\sigma_4} \otimes L_{\partial\sigma_5}$,

– bdy components ordered – note $\mathrm{Diff}(\coprod_k S^1)$ can permute S^1's.

or: $B_\sigma : L_{\partial_{in}\Sigma\sigma} \to L_{\partial_{out}\sigma}$ – (could affect orientations)

(but $L_{\partial\Sigma\sigma}$, etc will be expressed as $\otimes$'s of line bundles)

w/ props:

1). B_σ indep of ~~para~~ parametrization of Σ; | more careful defn's: next week

2) respects gluing of surfaces; :

vs $\Sigma_1 * \Sigma_2$. ($\xrightarrow{\sigma} X$).

holonomy analogue: closed surface Σ; cob. from $\emptyset$ to $\emptyset$.. ← shd'g (-note – $L_\emptyset$ a line;

– take $L_\emptyset$ = canonical copy of $\mathbb{C}$; get $B_{\Sigma\sigma} : \mathbb{C} \to \mathbb{C}$ linear

so $B_{\Sigma\sigma} \in \mathbb{C}^\times$; in fact, assume $B_{\Sigma\sigma} \in U(1)$.

hence: get map $B_\sigma : \mathrm{Emb}(\Sigma, \mathbb{R}^d) \times_{\mathrm{Diff}(\Sigma)} \mathrm{Map}(\Sigma, X) \to U(1)$. (~1992?) (ref: J-L Brylinski, Loop Spaces, char. classes & geom. quant.)

– curvature form: exists; describe via Chern-Weil thy on LX:

$H_B \in \Omega^3_{cl}(X; \mathbb{C})$; s.t.:

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1) $dH_B = 0$; & $[H_B] \in H^3(X;\mathbb{C})$ is given by: (S^1-transfer)

$LX \times S^1 \xrightarrow{\text{eval}} X$ defines $H^q(X) \to H^q(LX \times S^1) = (H^*LX \otimes_{\mathbb{C}} H^*S^1)_q$

$\xrightarrow{t}$ $\downarrow$

$H^{q-1}(LX) \otimes H^1(S^1)$ $\cong \mathbb{C}$

$= H^{q-1}(LX)$.

(related to xfer)

line bundle on LX

Then $t([H_B]) = c_1\left(\begin{smallmatrix} \mathcal{L}_B \\ \downarrow \\ LX \end{smallmatrix}\right)$;

- if $\exists$ 3-mfld Y^3, + map $\xrightarrow{\psi} X$, where $\partial Y^3 = \Sigma$, define $H_B(\psi)$

say $\displaystyle\int_{Y^3} \psi^* H_B$; then $e^{2\pi i H_B(\psi)} = B_{\partial\psi} \in \underline{U(1)}$.

+ M-W eqn analogues: (X, g, B) (metric, B-field) must satisfy string field eqn:

$S(X,g,B) = \displaystyle\int_X (R\,d\mathrm{vol} + H \wedge *H)$ - EM + gravity eqns, constant.

want (X,g,B) a critical pt. of this S fn'al.

get: $\longrightarrow$ Conformal field theory : ("string background")

1). $\mathbb{C}$-v.s. $\mathcal{H}$ (or Hilbert space, etc.) (top $\to$ TQFT).

2). + part operator: for each conformal surface Σ, have op $\mu_\Sigma : \mathcal{H}^{\otimes p} \to \mathcal{H}^{\otimes q}$,

Σ a cob. from p S^1's to q S^1's + gluing axioms. (forces alg. struct).

- construct from critical (X,g,B) :

$\mathcal{H} = L^2\left(\begin{smallmatrix} \mathcal{L}_B \\ \downarrow \\ LX \end{smallmatrix}\right)$; get μ_Σ : "integral op" : as follows:

Define $K : (LX)^p \times (LX)^q \to \mathbb{R}$ (or $\mathbb{C}$). ; $(\gamma_1, \cdots, \gamma_p) \times (\gamma_{p+1}, \cdots, \gamma_{p+q})$

$\mapsto \displaystyle\int e^{iS(\phi)}\, D\phi$ taken over $\phi : \Sigma \to X$ w/ $\partial\phi = \gamma_i$; ...

& $S(\phi) = E(\phi)$ (Dirichlet energy of ϕ) $+ i B_\phi$.

$\in H^{4q}$ $\quad$ $y \in LX^2$

- now define u_Σ: for $\alpha \in H^{*P}$, $u_\Sigma(\alpha) := u_\Sigma(\alpha)(y) = \int_{x \in \underline{LX}^0} k(x, y)\alpha(y)\, d\mu(x)$. ("path integral").

(*) not rigorous (*) . - really - $\underline{P\text{-}T}$ $\underline{\text{construction}}$ Σ .

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