The Going-Up Theorem

Throughout this discussion, we fix an integral ring extension $A \subset B$.

Theorem 1 *Suppose $P \subset A$ is a prime ideal. Then there exists a prime ideal $Q \subset B$ with $Q \cap A = P$.* □

Lemma 1 *If $J \subset B$ is an ideal and $J \cap A = I$, then $A/I \subset B/J$ is an integral ring extension.*

Proof An element $b \bmod J \in B/J$ satisfies the same monic polynomial over A/I that b satisfies over A. ■

The theorem and this first lemma combine to give the following result, which is sometimes called the Going Up Theorem. One just applies the theorem to $A/P_m \subset B/Q_m$.

Theorem 2 (Going-Up) *If $A \subset B$ is an integral ring extension and if $P_0 \subset P_1 \subset \cdots \subset P_n$ is a chain of prime ideals in A, and if $Q_0 \subset Q_1 \subset \cdots \subset Q_m$ is a chain of prime ideals of B with $Q_j \cap A = P_j$, $0 \leq j \leq m < n$, then the chain in B can be extended to $Q_m \subset Q_{m+1}$, with $Q_{m+1} \cap A = P_{m+1}$.* □

Lemma 2 *If $Q \subset B$ is an ideal and $Q \cap A = P$, then Q is maximal in B if and only if P is maximal in A.*

Proof If P is maximal then A/P is a field and every element of B/Q is algebraic over A/P. Hence B/Q is a field. Conversely, if Q is maximal and $x \neq 0 \in A/P$, then $1/x \in B/Q$ satisfies a monic polynomial over A/P, say $(1/x)^n + a_1(1/x)^{n-1} + \cdots + a_n = 0$. Multiply by x^n and solve for $1/x \in A/P$. Thus, A/P is a field. ■

Lemma 3 *If $S \subset A$ is a multiplicative set, then $S^{-1}A \subset S^{-1}B$ is an integral ring extension.*

Proof First of all, $S \subset A \subset B$, so S is a multiplicative subset of B and $S^{-1}B$ is defined. Next, even though $i_S : A \to S^{-1}A$ might not be injective, the natural map $S^{-1}A \to S^{-1}B$ is injective. Namely, if $[a/s] = 0 \in S^{-1}B$, then $as' = 0$ for some $s' \in S$ and hence $[a/s] = 0 \in S^{-1}A$. Finally, a monic degree n polynomial with coefficients in $S^{-1}A$ that has $[b/s] \in S^{-1}B$ as a root is obtained by dividing by s^n a monic degree n polynomial with coefficients in A that has b as a root. ■

Proof (Going-Up) Let $S = A - P$. Choose *any* maximal ideal $Q_S \subset S^{-1}B$. Then, by Lemma 2, $Q_S \cap S^{-1}A$ is a maximal ideal in the local ring $S^{-1}A$, hence $Q_S \cap S^{-1}A = P^e$, the unique maximal ideal of $S^{-1}A$. Now let $Q = Q_S^c = (j_S)^{-1}Q_S \subset B$, where $j_S : B \to S^{-1}B$ is the canonical map. Then $Q \subset B$ is a prime ideal and $Q \cap A = P$, since $P^{ec} = (i_S)^{-1}P^e = P \subset A$, where $i_S : A \to S^{-1}A$ is the canonical map.

(To follow the manipulations with the four prime ideals here, it helps to think in terms of the following diagram:

$$\begin{array}{ccc} A & \subset & B \\ {\scriptstyle i_S}\downarrow & & \downarrow{\scriptstyle j_S} \\ S^{-1}A & \subset & S^{-1}B \end{array}$$

One starts with a maximal ideal $Q_S \subset S^{-1}B$. Two contraction steps clockwise around the diagram takes you first to P^e, the unique maximal ideal of $S^{-1}A = A_{(P)}$, and then to $P \subset A$. Two contraction steps counterclockwise around the diagram takes you first to some prime ideal $Q \subset B$, which then must contract to $P \subset A$.) ■

Here is a final result about chains of prime ideals in an integral ring extension $A \subset B$.

Proposition 1 *If $Q \subset Q' \subset B$ are distinct prime ideals of B, then $P \subset P' \subset A$ are distinct prime ideals of A, where $P = Q \cap A$ and $P' = Q' \cap A$. In particular, if A and B are integral domains and if Q is a non-zero prime ideal of B, then $Q \cap A$ is a non-zero prime ideal of A.*

Proof The second statement actually implies the first, by looking at $P'/P = (Q' \cap A)/P \subset A/P \subset B/Q$. For the second statement, if $0 \neq x \in Q$, and if $f(x) = x^n + a_1x^{n-1} + \cdots + a_n$ is a monic polynomial over A of least degree with $f(x) = 0$, then $a_n \neq 0$, since B is an integral domain. But obviously $f(x) = 0$ implies $a_n \in Q \cap A$. ■